Optimal Interest-Rate Setting in a Dynamic IS/AS Model*

HENRIK JENSEN
Department of Economics
University of Copenhagen†

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†Address: Øster Farimagsgade 5, Building 26, Dk-1353 Copenhagen K, Denmark. E-mail: Henrik.Jensen@econ.ku.dk. Web: hjeconomics.dk.

Preface

This note deals with interest-rate setting in a simple dynamic macroeconomic setting. The purpose is to present some basic and central properties of an optimal interest-rate rule. The model framework predates the New-Keynesian paradigm of the late 1990s and onwards (it is accordingly dubbed "Old-Keynesian" by, e.g., Cochrane, 2011), but the simplicity of the framework allows clear-cut results to be stated, and fundamental properties of inflation-targeting regimes to be analyzed.

The exposition builds to some extent on Svensson (1997) and Walsh (2003, Section 10.4.2), and merges these presentations into one model.

Henrik Jensen, April 2011

1 Introduction

When setting the nominal interest rate, a central bank faces a number of issues. First of all, it must take into account its mandate for policymaking. Should it aim for stable prices? Stable employment? A combination of the two? Or maybe something else? Secondly, its decisions are difficult since information about relevant macroeconomic variables, which can be influenced by policy, are often not available at the time of policy implementation. As an example, most empirical evidence supports that inflation today to some extent is not controllable by a central bank; instead its actions will influence future inflation. Hence, expectations about future economic developments become central for a central bank taking actions today.

This note will settle these issues in a model that, despite its simplicity, captures some features of real-life economies, and results in a portrayal of optimal interest-rate setting, which resembles observed behavior by many central banks. In particular, the model can be used to highlight some basic properties of inflation targeting—the monetary policymaking regime that since the 1990s has been adopted by numerous central banks around the globe.

2 The Model

The model is a dynamic, log-linear IS/AS model. While it captures elements of models based on optimizing behavior, there is no explicit micro foundations, and one should be appropriately careful by using it to assess the performance of different policy regimes. However, the main purpose here is to spell out some basic features of optimal interest-rate setting, so the Lucas critique is ignored.

The demand side of the model is given by an "IS curve":

$$y_{t+1} = \theta y_t - \sigma^{-1}\left(i_t - \mathrm{E}_t \pi_{t+1}\right) + u_{t+1}, \qquad 0 < \theta < 1, \quad \sigma > 0, \tag{1}$$

where y_{t+1} is log of output, i_t is the nominal interest rate (the monetary policy instrument), π_{t+1} is the inflation rate, u_{t+1} is a mean-zero, serially uncorrelated shock. E_t is the rational-expectations operator conditional on information up to and including period t. The economy's supply side is characterized by a Phillips-curve relationship:

$$\pi_{t+1} = \pi_t + \kappa y_t + e_{t+1}, \qquad \kappa > 0, \tag{2}$$

where e_{t+1} is a mean-zero, serially uncorrelated shock.

The central aspect of the model is the timing. As seen from (1), any changes in the nominal interest rate affects output with a one-period lag. If the model is interpreted as quarterly, this does not appear unrealistic. Likewise, there is a delay between changes in demand in the economy and the inflation rate; cf. (2) where it is period-t output that affects period-$t+1$ inflation. Apart from these delayed effects of the nominal interest rate and demand, respectively, the equations have the usual interpretations.

The delays have important implications for the impact of a change in monetary policy in period t: It affects demand (output) one period ahead, and inflation (through the demand channel) two periods ahead. The feature that demand is affected first and inflation last, is in accordance with much VAR evidence, and is conventional wisdom within central banks. This clearly represents a challenge for the central bank in period t, as it does not see the shocks affecting output and inflation at the horizon it can actually affect these variables: u_{t+1}, e_{t+1}, e_{t+2} are unknown in period t.[1]

The policymaker, however is not helpless, as information about "where" the economy is today, reveals relevant information about the future. In period t, output and inflation are given by history, but due to the persistence embedded in equations (1) and (2), their current values provide information about their future values, which *can* be affected by monetary policy.

The metric for optimal policymaking will be the following utility function:

$$U = \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left[-\frac{\lambda}{2} y_{t+i}^2 - \frac{1}{2} \pi_{t+i}^2 \right], \qquad 0 < \beta < 1, \quad \lambda > 0. \tag{3}$$

This aims at representing a mandate under which the central bank should avoid variations in inflation (from an inflation goal normalized to zero), as well as variations in output. The parameter λ represents the relative weight attached to the latter objective. Note that the steady state of the model (since it is formulated in logs without constants, and with mean-zero shocks) will feature $y^{ss} = \pi^{ss} = 0$. Hence, we can interpret $y^{ss} = 0$ as the constant natural rate of output. Note that (3) penalizes deviations of log of output from zero; hence, there is no preference for output higher than the natural rate as in the Barro and Gordon-type models. As a consequence, the model will not feature an inflation bias, or other issues related to credibility. The focus is exclusively on optimal stabilization policy.

2.1 Stability analysis under an exogenous nominal interest rate

Before proceeding to the derivation of optimal policy, it is relevant to assess the model's stability properties for an exogenous nominal interest rate. For this purpose, the model is written in matrix form—a method which will be useful later also. To obtain a compact representation, it is convenient to note that by (2),

$$\mathrm{E}_t \pi_{t+1} = \pi_t + \kappa y_t. \tag{4}$$

Period-t expected inflation is a linear combination of the two predetermined variables y_t and π_t, making it predetermined as well. Given the assumed delays in policymaking, this

[1] Clearly, by the assumption of white-noise shocks, the informational disadvantage is taken to the extreme. The central message, however, applies with persistent shocks: There will be innovations to the processes, which are not known at the time of policy implementation.

makes sense, as policy in period t cannot affect inflation one period ahead. Consequently, $\mathrm{E}_t\pi_{t+1}$ is exogenous from the perspective of period t. Inserting (4) into (1) yields

$$y_{t+1} = \left(\theta + \sigma^{-1}\kappa\right) y_t + \sigma^{-1}\pi_t - \sigma^{-1} i_t + u_{t+1}. \tag{5}$$

Equations (2) and (5) provide the dynamics of output and inflation, which in matrix form becomes:

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} + \mathbf{B} i_t + \begin{bmatrix} u_{t+1} \\ e_{t+1} \end{bmatrix}, \tag{6}$$

where

$$\mathbf{A} \equiv \begin{bmatrix} \theta + \sigma^{-1}\kappa & \sigma^{-1} \\ \kappa & 1 \end{bmatrix}, \qquad \mathbf{B} \equiv \begin{bmatrix} -\sigma^{-1} \\ 0 \end{bmatrix}.$$

The dynamic system (6) has two predetermined state variables, y_t and π_t. When the nominal interest is considered exogenous, a unique non-explosive solution to the system exists if matrix $\mathbf{A}$ has two stable eigenvalues (or, characteristic roots). In other words, the eigenvalues should both be within the unit circle. On this, and requirements for stability in linear dynamics systems under rational expectations in general, the classical reference is Blanchard and Kahn (1980).

We can find the eigenvalues of $\mathbf{A}$ as the values of μ solving

$$\begin{vmatrix} \theta + \sigma^{-1}\kappa - \mu & \sigma^{-1} \\ \kappa & 1 - \mu \end{vmatrix} = 0.$$

The left-hand side determinant provides a second-order characteristic polynomial in μ:

$$p(\mu) = \mu^2 + a_1\mu + a_0,$$

with

$$a_1 = -\left(1 + \theta + \sigma^{-1}\kappa\right), \qquad a_0 = \theta. \tag{7}$$

This can be evaluated explicitly at $p(\mu) = 0$, and one can then examine the solutions to μ in detail. A simpler route, however, is to use a theorem by LaSalle (1986), as presented by Bullard and Mitra (2002), which states that a matrix $\mathbf{A}$ has both eigenvalues inside the unit circle if and only if

$$|a_0| < 1, \tag{8}$$

$$|a_1| < 1 + a_0, \tag{9}$$

both holds. Condition (8) holds as $0 < \theta < 1$. Condition (9), on the other hand, becomes $\sigma^{-1}\kappa < 0$ which does not hold.

For an exogenous path of the interest rate, the model is therefore not stable. The intuition is that if a shock increases output or inflation, this will lead to higher inflation in the next period. And the real interest rate will, in absence of any response from the nominal interest rate, fall, which causes further stimulus to output and eventually inflation. The economy will be on an explosive path.

The aim of optimal policymaking is therefore both to secure stability (stationarity) and to provide the appropriate stabilization of output and inflation. We now turn to the derivation of optimal interest-rate policy.

3 Optimal policy

We will adopt dynamic programming techniques to derive the optimal interest rate rule. For this purpose it proves appropriate, and convenient, to consider $\mathrm{E}_t\pi_{t+1}$ as the model's state variable, and $I_t \equiv \sigma^{-1}i_t - \theta y_t$ the policy instrument.[2] With these definitions (1) is rewritten as

$$y_{t+1} = \sigma^{-1}\mathrm{E}_t\pi_{t+1} - I_t + u_{t+1}. \tag{10}$$

Furthermore, (2) is forwarded one period (as π_{t+1} cannot be affected by i_t and therefore I_t) to $\pi_{t+2} = \pi_{t+1} + \kappa y_{t+1} + e_{t+2}$ which can be written in terms of the state as $\pi_{t+2} = \mathrm{E}_t\pi_{t+1} + e_{t+1} + \kappa y_{t+1} + e_{t+2}$, and eventually as a function of the policy instrument by inserting (10):

$$\pi_{t+2} = \left(1 + \sigma^{-1}\kappa\right)\mathrm{E}_t\pi_{t+1} - \kappa I_t + e_{t+1} + \kappa u_{t+1} + e_{t+2}. \tag{11}$$

These equations highlight that a policy contraction in period t (an increase in i_t and thus I_t), reduces output in period $t+1$ and inflation in period $t+2$.

3.1 The value function

To solve the optimization problem, we define the value function V as

$$V\left(\mathrm{E}_t\pi_{t+1}\right) \equiv \max \widetilde{U};$$

i.e., the maximum of

$$\widetilde{U} \equiv \beta^{-1}\left[U + \mathrm{E}_t\left\{\frac{\lambda}{2}y_t^2 + \frac{1}{2}\pi_t^2 + \frac{\beta}{2}\pi_{t+1}^2\right\}\right], \tag{12}$$

which is the part of utility that can be influenced by policy. Note that $\widetilde{U}$ is utility U from which the components involving y_t, π_t, π_{t+1} are subtracted. This is a valid transformation as these terms are independent of policy (note also that utility is scaled by β^{-1} for sake

[2] One *could* treat y_t and π_t as independent state variables and consider i_t as the policy instrument, but the solution will be the same. The approach taken here is much simpler (in particular since the value function will only depend on one state instead of two).

of simplicity). With $\widetilde{U}$ given by (12), the value function is given by

$$\begin{aligned} V\left(\mathrm{E}_t\pi_{t+1}\right) &= \max\ \mathrm{E}_t\left\{-\frac{\lambda}{2}y_{t+1}^2-\frac{\beta}{2}\pi_{t+2}^2-\frac{\beta\lambda}{2}y_{t+2}^2-\frac{\beta^2}{2}\pi_{t+3}^2-\frac{\beta^2\lambda}{2}y_{t+3}^2-\frac{\beta^3}{2}\pi_{t+4}^2-...\right\} \\ &= \max\ \mathrm{E}_t\left\{-\frac{\lambda}{2}y_{t+1}^2-\frac{\beta}{2}\pi_{t+2}^2+\beta\left[-\frac{\lambda}{2}y_{t+2}^2-\frac{\beta}{2}\pi_{t+3}^2-\frac{\beta\lambda}{2}y_{t+3}^2-\frac{\beta^2}{2}\pi_{t+4}^2-...\right]\right\} \\ &= \max\ \mathrm{E}_t\left\{-\frac{\lambda}{2}y_{t+1}^2-\frac{\beta}{2}\pi_{t+2}^2+\beta V\left(\mathrm{E}_{t+1}\pi_{t+2}\right)\right\} \qquad (13) \end{aligned}$$

where maximization is with respect to I_t, and subject to constraints (10) and (11).

3.2 Optimization

By inserting the constraints into the value function, one readily recover the first-order condition for optimal I_t:

$$\mathrm{E}_t\left\{\lambda y_{t+1}+\kappa\beta\pi_{t+2}-\beta\kappa V'\left(\mathrm{E}_{t+1}\pi_{t+2}\right)\right\}=0. \qquad (14)$$

We find the partial derivative of the value function (using the envelope theorem) as

$$V'\left(\mathrm{E}_t\pi_{t+1}\right)=\mathrm{E}_t\left\{-\lambda\sigma^{-1}y_{t+1}-\beta\left(1+\sigma^{-1}\kappa\right)\pi_{t+2}+\beta\left(1+\sigma^{-1}\kappa\right)V'\left(\mathrm{E}_{t+1}\pi_{t+2}\right)\right\} \qquad (15)$$

Forward (15) one period and take period-t expectations to obtain:

$$\mathrm{E}_tV'\left(\mathrm{E}_{t+1}\pi_{t+2}\right)=\mathrm{E}_t\left\{-\lambda\sigma^{-1}y_{t+2}-\beta\left(1+\sigma^{-1}\kappa\right)\pi_{t+3}+\beta\left(1+\sigma^{-1}\kappa\right)V'\left(\mathrm{E}_{t+2}\pi_{t+3}\right)\right\}.$$

This is inserted back into (14) to eliminate $\mathrm{E}_tV'\left(\mathrm{E}_{t+1}\pi_{t+2}\right)$:

$$\mathrm{E}_t\left\{\lambda y_{t+1}+\kappa\beta\pi_{t+2}+\beta\kappa\lambda\sigma^{-1}y_{t+2}+\beta^2\kappa\left(1+\sigma^{-1}\kappa\right)\pi_{t+3}-\beta^2\kappa\left(1+\sigma^{-1}\kappa\right)V'\left(\mathrm{E}_{t+2}\pi_{t+3}\right)\right\}=0. \qquad (16)$$

Forwarding the first-order condition one period, and taking period-t expectations, give $\mathrm{E}_t\left\{\lambda y_{t+2}+\kappa\beta\pi_{t+3}\right\}=\beta\kappa\mathrm{E}_t\left\{V'\left(\mathrm{E}_{t+2}\pi_{t+3}\right)\right\}$, which used in (16) gives

$$\mathrm{E}_t\left\{\lambda y_{t+1}+\kappa\beta\pi_{t+2}-\beta\lambda y_{t+2}\right\}=0. \qquad (17)$$

This optimality condition can now be combined with the model's equations in order to provide a characterization of optimal policy as portrayed by I_t.

First note that from (5) we find

$$y_{t+2}=\left(\theta+\sigma^{-1}\kappa\right)y_{t+1}+\sigma^{-1}\pi_{t+1}-\sigma^{-1}i_{t+1}+u_{t+2},$$

and therefore

$$y_{t+2}=\sigma^{-1}\kappa y_{t+1}+\sigma^{-1}\left(\mathrm{E}_t\pi_{t+1}+e_{t+1}\right)-I_{t+1}+u_{t+2}.$$

Inserting this, and the expressions for y_{t+1} and π_{t+2} [(10) and (11), respectively] into (17), yield

$$\begin{aligned} & \mathrm{E}_t\left\{\lambda\left(\sigma^{-1}\mathrm{E}_t\pi_{t+1}-I_t+u_{t+1}\right)\right\}+\mathrm{E}_t\left\{\kappa\beta\left[\left(1+\sigma^{-1}\kappa\right)\mathrm{E}_t\pi_{t+1}-\kappa I_t+e_{t+1}+\kappa u_{t+1}+e_{t+2}\right]\right\} \\ = \ & \mathrm{E}_t\left\{\beta\lambda\left(\sigma^{-1}\kappa y_{t+1}+\sigma^{-1}\left(\mathrm{E}_t\pi_{t+1}+e_{t+1}\right)-I_{t+1}+u_{t+2}\right)\right\}, \end{aligned}$$

and thus

$$\begin{aligned}
& \mathrm{E}_t\left\{\lambda\left[1-\beta\sigma^{-1}\kappa\right]\left(\sigma^{-1}\mathrm{E}_t\pi_{t+1}-I_t+u_{t+1}\right)\right\} \\
& +\kappa\beta\mathrm{E}_t\left\{\left[\left(1+\sigma^{-1}\kappa\right)\mathrm{E}_t\pi_{t+1}-\kappa I_t+e_{t+1}+\kappa u_{t+1}+e_{t+2}\right]\right\} \\
= \; & \beta\lambda\mathrm{E}_t\left\{\sigma^{-1}\left(\mathrm{E}_t\pi_{t+1}+e_{t+1}\right)-I_{t+1}+u_{t+2}\right\}.
\end{aligned}$$

With the simple assumptions about shocks, this reduces to

$$\begin{aligned}
& \lambda\left[1-\beta\sigma^{-1}\kappa\right]\left(\sigma^{-1}\mathrm{E}_t\pi_{t+1}-I_t\right)+\kappa\beta\left(1+\sigma^{-1}\kappa\right)\mathrm{E}_t\pi_{t+1}-\beta\kappa^2 I_t \\
= \; & \beta\lambda\left\{\sigma^{-1}\mathrm{E}_t\pi_{t+1}-\mathrm{E}_t I_{t+1}\right\},
\end{aligned}$$

which can be written as a first-order rational-expectations difference equation in I_t:

$$I_t=\left(1+\frac{\beta\left(\kappa\sigma-\lambda\right)}{\Psi}\right)\sigma^{-1}\mathrm{E}_t\pi_{t+1}+\frac{\beta\lambda}{\Psi}\mathrm{E}_t I_{t+1}, \tag{18}$$

with

$$\Psi\equiv\lambda\left[1-\beta\sigma^{-1}\kappa\right]+\beta\kappa^2.$$

We will assume throughout that $\sigma^{-1}\kappa<1$, such that $\Psi>0$ always holds. As we in our calibrations later on use the empirically plausible values $\kappa=0.1$ and $\sigma=2$, it should be clear that this is not a restrictive assumption.

In order to solve (18), we use the method of undetermined coefficients, and conjecture that the policy instrument I_t is a linear function of the state variable:

$$I_t=\phi\sigma^{-1}\mathrm{E}_t\pi_{t+1}. \tag{19}$$

Under this conjecture, $\mathrm{E}_t I_{t+1}=\phi\sigma^{-1}\mathrm{E}_t\pi_{t+2}=\phi\sigma^{-1}\left[\left(1+\sigma^{-1}\kappa\right)\mathrm{E}_t\pi_{t+1}-\kappa I_t\right]$, and therefore

$$\mathrm{E}_t I_{t+1}=\phi\sigma^{-1}\left[\left(1+\sigma^{-1}\kappa\right)-\kappa\phi\sigma^{-1}\right]\mathrm{E}_t\pi_{t+1}. \tag{20}$$

Inserting (19) and (20) into the difference equation (18) yields:

$$\phi\sigma^{-1}\mathrm{E}_t\pi_{t+1}=\left(1+\frac{\beta\left(\kappa\sigma-\lambda\right)}{\Psi}\right)\sigma^{-1}\mathrm{E}_t\pi_{t+1}+\frac{\beta\lambda}{\Psi}\left[\phi\sigma^{-1}\left(1+\sigma^{-1}\kappa\right)-\kappa\phi^2\sigma^{-2}\right]\mathrm{E}_t\pi_{t+1}.$$

This verifies the form of the conjecture (19), and identifies the unknown policy-rule coefficient as the solution to

$$\phi=1+\frac{\beta\left(\kappa\sigma-\lambda\right)}{\Psi}+\frac{\beta\lambda}{\Psi}\left[\phi\left(1+\sigma^{-1}\kappa\right)-\kappa\phi^2\sigma^{-1}\right],$$

which is a second-order polynomial in ϕ:

$$\beta\lambda\kappa\sigma^{-1}\phi^2+\left(\Psi-\beta\lambda\left(1+\sigma^{-1}\kappa\right)\right)\phi-\left(\Psi+\beta\left(\kappa\sigma-\lambda\right)\right)=0. \tag{21}$$

The solutions to (21) are

$$\phi=\frac{-\left(\Psi-\beta\lambda\left(1+\sigma^{-1}\kappa\right)\right)\pm\sqrt{\left(\Psi-\beta\lambda\left(1+\sigma^{-1}\kappa\right)\right)^2+4\beta\lambda\kappa\sigma^{-1}\left(\Psi+\beta\left(\kappa\sigma-\lambda\right)\right)}}{2\beta\lambda\kappa\sigma^{-1}}. \tag{22}$$

To assess whether the high or low solution of ϕ is relevant, we first assess stability of the model for *any* value of ϕ.

3.3 Stability analysis under the policy rule for any ϕ

With $I_t = \phi\sigma^{-1}\mathrm{E}_t\pi_{t+1}$, we have that $\sigma^{-1}i_t - \theta y_t = \phi\sigma^{-1}\mathrm{E}_t\pi_{t+1} = \phi\sigma^{-1}(\pi_t + \kappa y_t)$ and therefore

$$i_t = \sigma\theta y_t + \phi\mathrm{E}_t\pi_{t+1}. \tag{23}$$

We can then rewrite the dynamic system in matrix form by using (23) and the fact that $\mathrm{E}_t\pi_{t+1} = \pi_t + \kappa y_t$, along with (6):

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} = \mathbf{A}\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} + \mathbf{B}\left(\left[\sigma\theta + \phi\kappa\right] y_t + \phi\pi_t\right) + \begin{bmatrix} u_{t+1} \\ e_{t+1} \end{bmatrix},$$

or,

$$\begin{aligned} \begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} &= \mathbf{A}\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} + \widetilde{\mathbf{B}}\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} u_{t+1} \\ e_{t+1} \end{bmatrix}, \\ \widetilde{\mathbf{B}} &\equiv \begin{bmatrix} -\theta - \sigma^{-1}\phi\kappa & -\sigma^{-1}\phi \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{24}$$

Stability of the system (24) is then secured when the matrix

$$\widetilde{\mathbf{A}} \equiv \mathbf{A} + \widetilde{\mathbf{B}} = \begin{bmatrix} \sigma^{-1}\kappa(1-\phi) & \sigma^{-1}(1-\phi) \\ \kappa & 1 \end{bmatrix}$$

has two eigenvalues within the unit circle. The eigenvalues of $\widetilde{\mathbf{A}}$ are the values of μ solving

$$\begin{vmatrix} \sigma^{-1}\kappa(1-\phi) - \mu & \sigma^{-1}(1-\phi) \\ \kappa & 1-\mu \end{vmatrix} = 0.$$

The left-hand side determinant provides a second-order characteristic polynomial in μ:

$$\widetilde{p}(\mu) = \mu^2 + a_1\mu + a_0,$$

with

$$a_1 \equiv -\left[1 + \sigma^{-1}\kappa(1-\phi)\right], \qquad a_0 \equiv 0. \tag{25}$$

We can then evaluate whether the necessary and sufficient conditions for two stable roots, (8) and (9), are satisfied. Condition (8), $|a_0| < 1$, is clearly satisfied whereas condition (9), when (25) applies, becomes $|1 + \sigma^{-1}\kappa(1-\phi)| < 1$. A necessary condition for this to hold is $\phi > 1$. Hence, the relevant solution to (21) is the large root, as long as this is greater than one.[3] In Appendix A, we show that the large root of (22) indeed is larger than one.

Hence, optimal interest-rate setting in this dynamic economy is $i_t = \sigma\theta y_t + \phi\mathrm{E}_t\pi_{t+1}$, where $\phi > 1$ is given by the larger root of (22).

[3] The relevance of a root $\phi > 1$ in terms of securing stability can also be seen by inserting $I_t = \sigma^{-1}\phi\mathrm{E}_t\pi_{t+1}$ into (11), which yields $\pi_{t+2} = \left[1 + \sigma^{-1}\kappa(1-\phi)\right]\mathrm{E}_t\pi_{t+1} + e_{t+1} + \kappa u_{t+1} + e_{t+2}$, and therefore $\pi_{t+2} = \left[1 + \sigma^{-1}\kappa(1-\phi)\right]\pi_{t+1} - \sigma^{-1}\kappa(1-\phi)e_{t+1} + \kappa u_{t+1} + e_{t+2}$. It is evident that stability requires $\left|1 + \sigma^{-1}\kappa(1-\phi)\right| < 1$, and thus $\phi > 1$. It also implies that ϕ cannot be too high, more specifically, $\phi < 1 + 2\sigma/\kappa$ must hold as well. For our benchmark calibration mentioned in Footnote 4, this latter condition is $\phi < 41$, and thus well within what is economically relevant.

4 Discussion

Several properties of the optimal rule are worth mentioning. It is of the same form of the policy rule proposed by Taylor (1993), in the sense that the nominal interest rate responds to output and inflation. Also, as is the case with the Taylor rule, the coefficient on inflation, ϕ is larger than one. This property is often referred to as the Taylor principle, and it is central for stability in this type of model economy. A positive shock to inflation or output will lead to increased inflation and inflation expectations. When the nominal interest rate raises by more than one-for-one with inflation, the real interest rate increases, which puts downward pressure on demand, and consequently on inflation. Figures 1 and 2 show the impulse response patterns for the economy following a unit shock to demand and supply, respectively.[4]

As evident from the figures, the optimal interest-rate policy plays an active role in bringing the economy back to steady state after either shock. In the case of a positive shock to demand, the interest rate is raised immediately to counteract the expansionary and inflationary effects that are anticipated in the future. As a result, output is undershooting from period 2 and onwards to dampen the inflationary effects. In the case of an inflationary supply shock, the nominal interest rate is again increased immediately, but much more fiercely, as inflation is driven up stronger and more persistently in comparison with a demand shock.

While the optimal rule shares qualitative features with the original Taylor rule, it differs in quantitative respects. Since the coefficients on output and inflation in the optimal rule are functions of the deep parameters of the model, their values can differ vastly from those in Taylor's rule. The original Taylor rule is in this notation given by

$$i_t = 0.5y_t + 1.5\pi_t,$$

whereas the optimal rule under the benchmark calibration (see Footnote 4) is given by $i_t = 1.0y_t + 3.5\mathrm{E}_t\pi_{t+1}$, which by use of (4) yields

$$i_t = 1.35y_t + 3.5\pi_t$$

Hence, a stronger response to both output and inflation is envisaged by the optimal rule for this calibration of the model. Many have estimated policy rules of the Taylor type for the Federal Reserve and the ECB, *inter alia*, and some find Taylor-rule-type behavior in data, while some do not (see, Cochrane, 2011, for recent discussions on Taylor-rule estimations, and Jensen and Aastrup, 2010, for examples of estimations on Euro-area data). Irrespective of what is exactly found empirically, it is a common finding that predictions from calibrations call for stronger policy reactions under optimal policy than

[4] These simulations are made with a benchmark parameterization with $\sigma = 2$, $\kappa = 0.1$, $\beta = 0.99$, $\theta = 0.5$, and $\lambda = 0.5$.

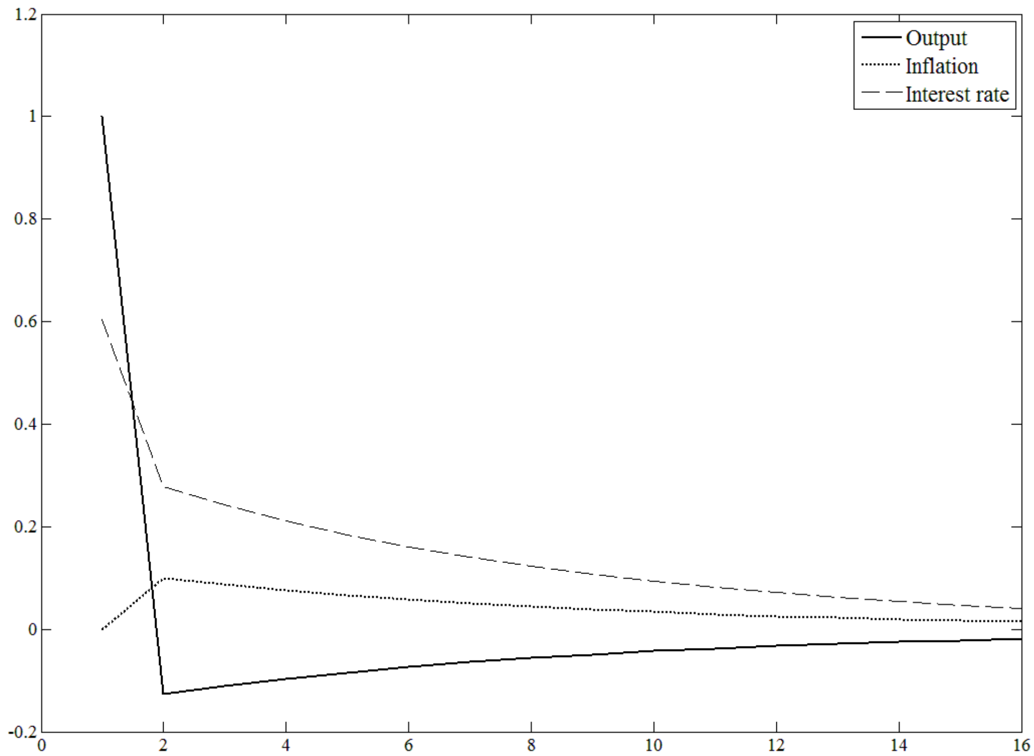

Figure 1: Impulse responses following a demand shock, $u_1 = 1$

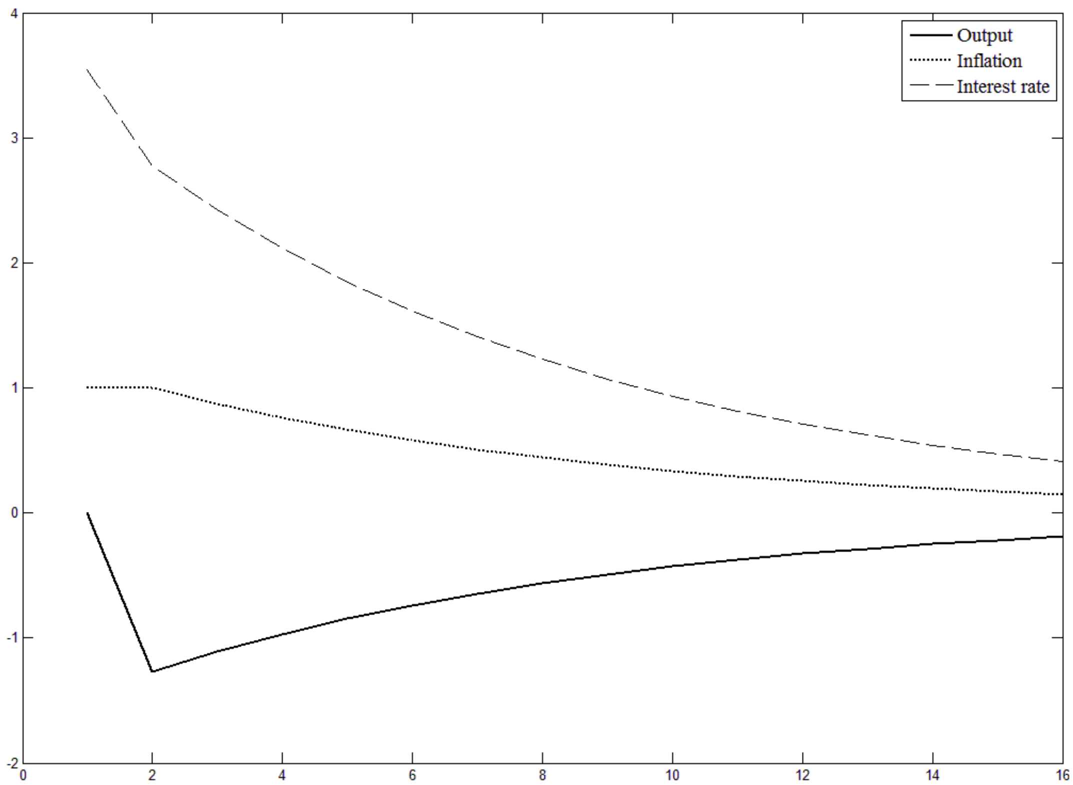

Figure 2: Impulse reponses following a supply shock, $e_1 = 1$

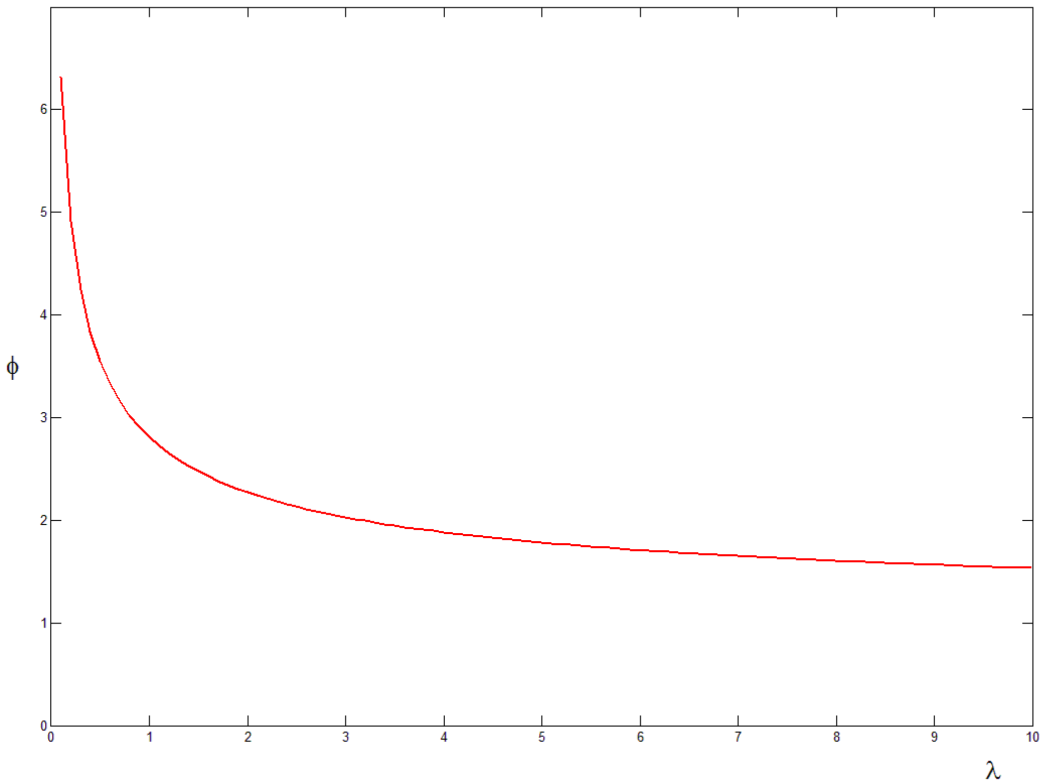

Figure 3: Optimal policy-rule coefficient and preferences

what is found in data. This could reflect a caution in real-life policymaking that the optimization exercise does not take into account.

Despite the high absolute values of the optimal coefficients, it is a general insight that the magnitudes need not reflect anything about the central bank's relative preference for output versus inflation stability. A relatively high weight on output, e.g., does not necessarily reflect a high preference for output stability, but could either be due to a high value of θ or a high value of σ. In either case, it will mean that the interest rate will have to be adjusted stronger to obtain the same effect on next period's output. The relative preference for output stability, however, does affect the value of the inflation coefficient—and in the expected direction. The higher is λ, the lower is ϕ, which is illustrated in Figure 3 where λ varies between 0.1 and 10 (with the remainder parameters equal to the benchmark calibration). This results in a variation in ϕ between 6 and 1.5.

In conclusion, observing different policy-rule parameters across countries does not necessarily reflect differences in preferences. It could, but it can also reflect different structural characteristics, which necessitate different interest-rate responses to fulfill identical objectives. It is in any case noteworthy that under optimal policymaking, the central bank responds to output even though is hardly cares about output stability. The reason is that with the assumed delays in policymaking, observed changes in output are relevant indicators for future inflation developments. Hence, the model under optimal policymaking provides an example of output being an *intermediate target* for a strongly anti-inflationary central bank.

5 Application: Inflation targeting

Inflation targeting is a monetary policy regime which differs very much in set-up across the countries who have adopted it. One thing is common though, namely the specification of some numerical target value (or target range) for the inflation rate, and some specification of the horizon at which the central bank intends to attain the target. In one of the first theoretical contributions, Svensson (1997) used a variant of the model of this note, to flesh out some central policy properties of inflation targeting under the two "variants" it can be viewed. Svensson (2010) contains an updated survey of the inflation-targeting literature.

5.1 "Strict" inflation targeting

Under "strict" inflation targeting, the central bank has the inflation target as the overriding objective for monetary policy. This will in context of this model correspond to the case of $\lambda = 0$. While many inflation-targeting countries indeed consider inflation stability as a top priority, several have caveats in their legal frameworks that allow some focus on output stabilization—as long as it does not interfere with the price stability objective. Nevertheless, the "strict" inflation target has a couple of clear implications. Assuming optimal policymaking, the optimality condition (17), often labelled a "targeting rule" in the inflation-targeting literature, will become:

$$\mathrm{E}_t\left\{\pi_{t+2}\right\} = 0. \tag{26}$$

In words, the central bank engages in *inflation forecast targeting.* By this is meant that it sets the interest rate such that the inflation rate at the horizon controllable by policy (here, two periods) is expected to be on target (here normalized to zero).

This leads to simple implications for policy implementation as well as policy evaluation. Concerning policy implementation a simple guideline would be that if the inflation forecast is over (under) target for an unchanged interest rate, then the central bank should raise (lower) the nominal interest rate. The precise interest rate rule can under strict inflation targeting be derived from (11):

$$\begin{aligned}
\mathrm{E}_t\pi_{t+2} &= \left(1+\sigma^{-1}\kappa\right)\mathrm{E}_t\pi_{t+1} - \kappa I_t \\
&= \left(1+\sigma^{-1}\kappa\right)\left(\pi_t + \kappa y_t\right) - \kappa\left(\sigma^{-1}i_t - \theta y_t\right),
\end{aligned}$$

where it is seen how changes in expected inflation two-periods ahead (caused by, e.g., changes in y_t or π_t) must be met by interest rate changes as explained above. Note that $\mathrm{E}_t\pi_{t+2} = 0$ is attained when

$$i = \left(1+\frac{\sigma}{\kappa}\right)\pi_t + \left(\kappa+\sigma+\sigma\theta\right)y_t.$$

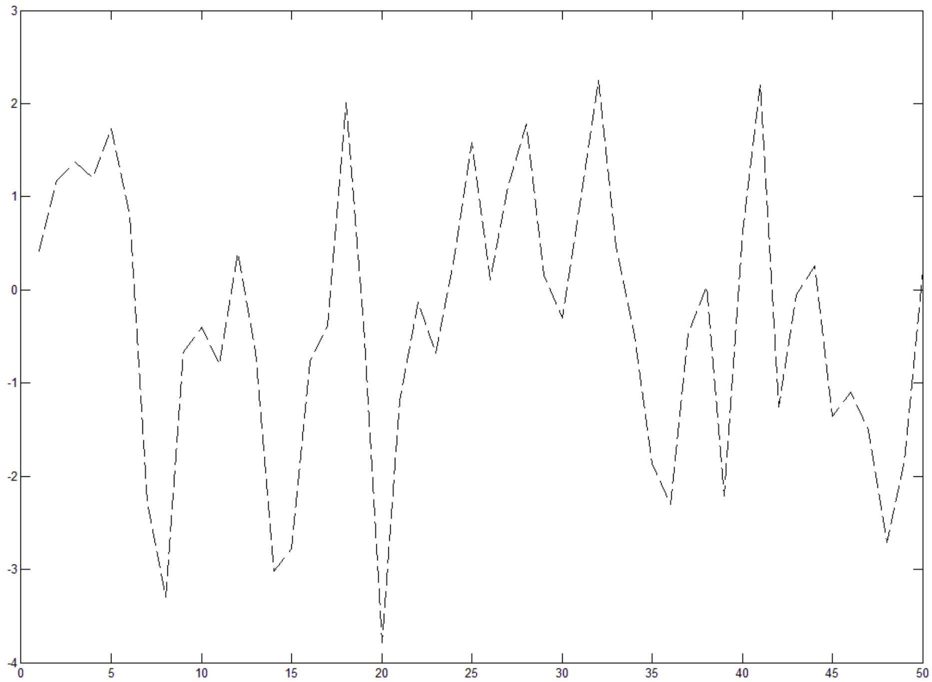

Figure 4: Stochastic simulation of actual inflation under "strict" (zero-) inflation targeting

In terms of evaluation of policy performance, the "strict" inflation targeting framework emphasizes clearly that one cannot judge the success of policy on the behavior of inflation. With the (realistic) delays in policy impact, shocks constantly bring inflation away from target. This is unavoidable even for an optimizing central bank only focusing on inflation stability. Figure 4 illustrates this. It shows a sample path for inflation in a simulation of the model under "strict" inflation targeting, where the demand and supply shocks in each period are drawn from independent $\mathcal{N}(0,1)$ distributions. It is seen that inflation fluctuates quite substantially, even though the central bank is doing the best possible job. Hence, an inflation targeting bank should not be judged on actual inflation, but on its ability to get the inflation forecast at the target value at the relevant horizon.

5.2 "Flexible" inflation targeting

Under "flexible" inflation targeting, $\lambda > 0$, and the central bank explicitly takes notice of output volatility. The main difference with this more realistic scenario and "strict" inflation targeting, is that the size of λ will affect the horizon at which the bank expects to have its inflation forecast match the target. We saw in Figure 3 that a higher λ resulted in a lower ϕ, and in Footnote 3 we saw that the equilibrium process for inflation is given by

$$\pi_{t+2} = \left[1 + \sigma^{-1}\kappa\left(1-\phi\right)\right]\pi_{t+1} - \sigma^{-1}\kappa\left(1-\phi\right)e_{t+1} + \kappa u_{t+1} + e_{t+2},$$

which is more persistent with the lower is ϕ. Hence, the inflation target will be reached later, the larger is λ. Central banks allowing longer horizons for the target to be met, can therefore be interpreted as central banks with relative high values of λ.

A Proof that the high root of (22) is larger than one

We start by rewriting (22) as

$$\begin{aligned}\phi &= \frac{-\left(\Psi-\beta\lambda\left(1+\sigma^{-1}\kappa\right)\right)}{2\beta\lambda\kappa\sigma^{-1}} \\ &\quad \pm\frac{\sqrt{\Psi^2+\beta^2\lambda^2\left(1+\sigma^{-1}\kappa\right)^2-2\Psi\beta\lambda\left(1+\sigma^{-1}\kappa\right)+4\beta\lambda\kappa\sigma^{-1}\left(\Psi+\beta\left(\kappa\sigma-\lambda\right)\right)}}{2\beta\lambda\kappa\sigma^{-1}},\end{aligned}$$

and thus

$$\begin{aligned}\phi &= \frac{-\left(\Psi-\beta\lambda\left(1+\sigma^{-1}\kappa\right)\right)}{2\beta\lambda\kappa\sigma^{-1}} \\ &\quad \pm\frac{\sqrt{\Psi^2+\beta^2\lambda^2\left(1+\sigma^{-1}\kappa\right)^2-2\Psi\beta\lambda+2\beta\lambda\kappa\sigma^{-1}\Psi+4\beta\lambda\kappa\sigma^{-1}\beta\left(\kappa\sigma-\lambda\right)}}{2\beta\lambda\kappa\sigma^{-1}}.\end{aligned}$$

The large root is larger than one if

$$\begin{aligned}&\frac{-\left(\Psi-\beta\lambda\left(1+\sigma^{-1}\kappa\right)\right)+\sqrt{\Psi^2+\beta^2\lambda^2\left(1+\sigma^{-1}\kappa\right)^2-2\Psi\beta\lambda+2\beta\lambda\kappa\sigma^{-1}\Psi+4\beta\lambda\kappa\sigma^{-1}\beta\left(\kappa\sigma-\lambda\right)}}{2\beta\lambda\kappa\sigma^{-1}} \\ &> 1,\end{aligned}$$

or

$$\begin{aligned}&-\left(\Psi-\beta\lambda\left(1+\sigma^{-1}\kappa\right)\right) \\ &+\sqrt{\Psi^2+\beta^2\lambda^2\left(1+\sigma^{-1}\kappa\right)^2-2\Psi\beta\lambda+2\beta\lambda\kappa\sigma^{-1}\Psi+4\beta\lambda\kappa\sigma^{-1}\beta\left(\kappa\sigma-\lambda\right)} \\ &> 2\beta\lambda\kappa\sigma^{-1}.\end{aligned}$$

This inequality is equivalent to

$$\begin{aligned}&\sqrt{\Psi^2+\beta^2\lambda^2\left(1+\sigma^{-1}\kappa\right)^2-2\Psi\beta\lambda+2\beta\lambda\kappa\sigma^{-1}\Psi+4\beta\lambda\kappa\sigma^{-1}\beta\left(\kappa\sigma-\lambda\right)} \\ &> 2\beta\lambda\kappa\sigma^{-1}+\left(\Psi-\beta\lambda\left(1+\sigma^{-1}\kappa\right)\right),\end{aligned}$$

$$\begin{aligned}&\Psi^2+\beta^2\lambda^2\left(1+\sigma^{-1}\kappa\right)^2-2\Psi\beta\lambda+2\beta\lambda\kappa\sigma^{-1}\Psi+4\beta\lambda\kappa\sigma^{-1}\beta\left(\kappa\sigma-\lambda\right) \\ &> \left[2\beta\lambda\kappa\sigma^{-1}+\left(\Psi-\beta\lambda\left(1+\sigma^{-1}\kappa\right)\right)\right]^2,\end{aligned}$$

$$\begin{aligned}&\Psi^2+\beta^2\lambda^2\left(1+\sigma^{-1}\kappa\right)^2-2\Psi\beta\lambda+2\beta\lambda\kappa\sigma^{-1}\Psi+4\beta\lambda\kappa\sigma^{-1}\beta\left(\kappa\sigma-\lambda\right) \\ &> 4\beta^2\lambda^2\kappa^2\sigma^{-2}+\left(\Psi-\beta\lambda\left(1+\sigma^{-1}\kappa\right)\right)^2+4\beta\lambda\kappa\sigma^{-1}\left(\Psi-\beta\lambda\left(1+\sigma^{-1}\kappa\right)\right),\end{aligned}$$

$$
\begin{aligned}
& \Psi^2+\beta^2\lambda^2\left(1+\sigma^{-1}\kappa\right)^2-2\Psi\beta\lambda+2\beta\lambda\kappa\sigma^{-1}\Psi+4\beta\lambda\kappa\sigma^{-1}\beta\left(\kappa\sigma-\lambda\right) \\
> \ & 4\beta^2\lambda^2\kappa^2\sigma^{-2}+\Psi^2+\beta^2\lambda^2\left(1+\sigma^{-1}\kappa\right)^2-2\Psi\beta\lambda\left(1+\sigma^{-1}\kappa\right)+4\beta\lambda\kappa\sigma^{-1}\left(\Psi-\beta\lambda\left(1+\sigma^{-1}\kappa\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
& -2\Psi\beta\lambda+2\beta\lambda\kappa\sigma^{-1}\Psi+4\beta\lambda\kappa\sigma^{-1}\beta\left(\kappa\sigma-\lambda\right) \\
> \ & 4\beta^2\lambda^2\kappa^2\sigma^{-2}-2\Psi\beta\lambda\left(1+\sigma^{-1}\kappa\right)+4\beta\lambda\kappa\sigma^{-1}\left(\Psi-\beta\lambda\left(1+\sigma^{-1}\kappa\right)\right),
\end{aligned}
$$

$$4\beta\lambda\kappa\sigma^{-1}\Psi+4\beta\lambda\kappa\sigma^{-1}\beta\left(\kappa\sigma-\lambda\right)>4\beta^2\lambda^2\kappa^2\sigma^{-2}+4\beta\lambda\kappa\sigma^{-1}\left(\Psi-\beta\lambda\left(1+\sigma^{-1}\kappa\right)\right),$$

$$\beta\lambda\kappa\sigma^{-1}\beta\left(\kappa\sigma-\lambda\right)>\beta^2\lambda^2\kappa^2\sigma^{-2}-\beta\lambda\kappa\sigma^{-1}\beta\lambda\left(1+\sigma^{-1}\kappa\right),$$

$$\kappa\sigma-\lambda>\lambda\kappa\sigma^{-1}-\lambda\left(1+\sigma^{-1}\kappa\right),$$

and finally

$$\kappa\sigma>0.$$

This always holds. Hence, for the high root ϕ, it follows that $\phi>1$.

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