Kinks and Gains from Credit Cycles*

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Abstract

We assess the welfare cost of business cycles in a calibrated small-open-economy model incorporating collateralized household borrowing. Business fluctuations impact credit tightness, leading to periods when households are financially unconstrained relative to a steady state characterized by a binding collateral constraint. The resulting nonlinearity in debt determination and its influence on households' consumption and saving decisions are pivotal to the emergence of a welfare *gain* from business cycles that overcomes conventional losses associated with uncertainty. As shocks become larger, households engage in precautionary saving to mitigate the risk of hitting their borrowing limit. Consequently, we observe lower average debt and higher average consumption, which results in a gain from business fluctuations. Analyzing the impact of pecuniary externalities in isolation, considering an alternative timing for the price of the collateral asset, or assuming the collateral constraint not to be binding in the steady state, does not significantly alter these fundamental properties.

Keywords: Cost of business cycles, collateral constraints, precautionary saving.

JEL codes: E20, E32, E66.

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1 Introduction

In this paper, we quantify the welfare cost of business cycles through the lens of a model featuring collateralized household borrowing. This is a relevant task, mainly for two reasons. First, from a practical viewpoint, the last two decades have witnessed a formidable resurgence of interest in the role of credit-market frictions for business fluctuations. Likewise, a significant concern on the policy agenda is to come up with macroprudential policies capable of dampening fluctuations that arise from or are magnified by financial markets. Second, from a modeling viewpoint, collateral constraints are key to generating sizeable asymmetries in economic activity, exacerbating cyclical downturns and smoothing upturns by tightening or relaxing credit conditions (see, e.g., Mendoza, 2010; Jensen *et al.*, 2020). Our key contribution is to show that, despite their fundamental role as business-cycle amplifiers, collateral constraints induce nonlinearities in households' economic decision rules that have the potential to make business fluctuations *welfare-improving.*

To understand this result, which may seem counterintuitive at first, it is convenient to begin by recalling the key insight of Lucas (1987). He started from a concave and continuous utility function, to then compare the outcome from the deterministic case with its counterpart in the stochastic case, where the state fluctuates around the same deterministic value. By Jensen's inequality, the former outcome is preferred, as a result of which some compensation would be necessary to make consumers indifferent between the two consumption paths. Using a conventional CRRA utility function and a statistical model of consumption, Lucas (1987) computed a welfare cost of business cycles as low as 0.008% of consumption.[1] This mechanism—which in the remainder we refer to as the *fluctuations* effect, following Cho *et al.* (2015)—is operative in the economy we envisage, as we employ a standard utility function exhibiting prudence (Kimball, 1990). We nest this in a calibrated small-open representative-agent economy where borrowing by domestic households is subject to a collateral constraint, with the underlying collateral asset being represented by the available stock of durables. Domestic households are assumed to be more impatient than international lenders, and therefore prone to borrowing. We impose

[1]This negligible size has since been contested by a vast literature identifying higher welfare costs of fluctuations in extensions of Lucas's simple framework. These include, but are not limited to, models with incomplete financial markets (İmrohoroğlu, 1989; Krusell and Smith, 1999; Krusell *et al.*, 2009; Storesletten *et al.*, 2001), imperfect competition (Galí *et al.*, 2007), detailed time-series modeling of consumption (Reis, 2007; De Santis, 2007), non-expected utility functions (Obstfeld, 1994), asset prices (Alvarez and Jermann, 2004), endogenous growth (Barlevy, 2004), and disaster risk (Barro, 2006).

the steady state of the model to be characterized by a binding collateral constraint.

In this setting, business cycles can be *beneficial* for welfare, the benefit being more than one order of magnitude larger than Lucas's number, in absolute value: Fluctuations resulting from shocks to credit limits and household income *increase* unconditional welfare by around 0.21% of quarterly consumption in perpetuity, according to our baseline calibration. In fact, we show that higher uncertainty typically increases the welfare gain. To see why this is the case, it is important to recognize that, at any point in time, the collateral constraint can be either binding—the so-called constrained regime, in which households borrow up to the available credit limit—or nonbinding—the unconstrained regime, in which case the constraint holds as a strict inequality. The resulting policy functions feature a *kink* at the point where the model switches from one regime to the other. An additional effect on welfare can therefore be detected, which we label *endogenous switching*: The presence of an occasionally nonbinding collateral constraint gives rise to a precautionary motive that complements prudence, as households engage in self-insurance to mitigate the risk of hitting against the borrowing limit. Thus, they respond to increased uncertainty by contracting less debt, which enables them to enjoy higher average consumption. At a realistic calibration of the model economy and for various model configurations, the endogenous switching effect is more potent than the fluctuations effect, thus paving the way for business cycles to entail a welfare gain.

Our main finding is robust to a range of alternative model assumptions that have been examined in previous contributions dealing with collateralized borrowing. A prominent strand of this literature has sought to understand the role of pecuniary externalities—the failure of individual households to internalize the general equilibrium effects of their borrowing decisions on asset prices—for welfare (see, e.g., Lorenzoni, 2008; and Bianchi, 2011, among others). In fact, we show that welfare in the baseline decentralized equilibrium (DE) is equivalent to that obtained by a social planner that internalizes the externality by attaining the constrained-efficient equilibrium (CEE). This result, which has first been pointed out by Ottonello *et al.* (2022), crucially depends on the collateral constraint featuring the expected *future* price of the collateral asset, in the baseline setting. In this case, the shadow value of borrowing in the DE is a rescaled version of that in the CEE, implying that no macroprudential policy is desirable. In line with Ottonello *et al.* (2022), we show that this equivalence breaks down when contemplating the *contemporaneous* price of the collateral asset, in which case welfare in the CEE dominates that in the DE. However, we obtain a welfare gain from business fluctuations regardless of the timing of the collateral constraint and of the type of equilibrium being considered.

While the assumption of a binding borrowing constraint in the steady state is in line with several studies (see, e.g., Kiyotaki and Moore, 1997; or Bianchi, 2011), another prominent strand of the literature assumes that households are financially unconstrained in the steady state (see, e.g., Mendoza, 2010). Our main finding transcends this divide, as we confirm the presence of a welfare gain when solving a version of the model featuring an unconstrained steady state. Essentially, the intuition remains consistent, relative to our baseline analysis: Uncertainty prompts households to increase their precautionary savings so as to be financially unconstrained as frequently as possible. This results in reduced average debt, elevated average consumption, and enhanced welfare.

The hypothesis that business fluctuations could be welfare-enhancing has rarely been put forward in the existing literature. An important exception is represented by Cho *et al.* (2015), who report that gains from business cycles may arise in a conventional real business cycle economy. The presence of multiplicative shocks is crucial in their setting: Such shocks have the potential to raise the mean level of output and/or consumption, allowing agents to make purposeful use of uncertainty by working harder and investing more during expansionary periods.[2] When uncertainty enters the economy additively, instead, it has no beneficial effect on the choices that can be adjusted to it. In the context of our model, we show that welfare gains may emerge not only in the presence of multiplicative credit-limit shocks, but also in response to additive income shocks alone. In fact, both types of perturbation have the potential to trigger endogenous switching, thus exerting an impact on the mean levels of debt and consumption in the presence of an occasionally nonbinding borrowing constraint. In this sense, the endogenous switching effect is complementary to the mean effect of Cho *et al.* (2015).

Finally, our paper is also related to recent work by Jordà *et al.* (2020), who observe a significant increase in the welfare cost of fluctuations, using a statistical model of consumption that matches the negative skewness of consumption-growth data. While we match an analogous data moment, our welfare analysis employs a structural framework, and is robust to assuming either a distorted or an undistorted steady state.

[2] As discussed by Fernández-Villaverde and Guerrón-Quintana (2020), the welfare gain in Cho *et al.* (2015) can be thought of as a household-side version of the Oi-Hartmann-Abel effect, according to which firms can expand and contract production in response to positive and negative shocks, so as to take advantage of a mean-preserving spread to raise average output (see Oi, 1961; Hartmann, 1972; and Abel, 1983). It is worth pointing out that Cho *et al.* (2015) derive their result in an economy in which equilibrium outcomes are Pareto efficient, both with and without shocks. In contrast, we first derive our baseline result in a setting where the steady state of the economy is inefficient, and fluctuations may entail temporary switches to an efficient regime, and then show that the result is confirmed in the opposite situation.

The findings we present have important implications for assessing and conducting economic stabilization policies. First, welfare gains from uncertainty may be observed even in settings where a social planner neutralizes pecuniary externalities, thus highlighting the essential role that occasionally binding financial constraints play in shaping business-cycle dynamics, irrespective of the institutional setting. Second, we add to the insights of Ottonello *et al.* (2022), who emphasize differences in the macroprudential policy outcomes of models featuring current- versus (expected) future-price constraints. While we confirm that such distinction entails very different implications regarding the role of pecuniary externalities, we indicate that welfare gains from uncertainty can emerge in either case.

The outline of the paper is as follows. Section 2 introduces the model; Section 3 describes the solution method; and Section 4 presents the calibration. Section 5 discusses the main result of the paper from a quantitative viewpoint, relying on different welfare criteria; Section 6 dissects the gain from business fluctuations through a number of exercises aimed at understanding its origins, examining the specific contribution of different shocks at play in the model, and evaluating the role of pecuniary externalities through a comparison between the DE and the CEE. Finally, Section 7 concludes. Various technical details and additional supplementary material are reported in the Appendix.

2 The model

We consider a small open economy with free capital mobility. Time is discrete, $t = 1, 2, ..., \infty$. The economy is inhabited by a continuum of homogeneous households of size 1 with utility:

$$U = \mathbb{E}_1 \left[\sum_{t=1}^{\infty} \beta^{t-1} \left(\frac{1}{1-\gamma} c_t^{1-\gamma} + \frac{\nu}{1-\gamma_h} h_t^{1-\gamma_h} \right) \right], \tag{1}$$

where c_t is consumption of a perishable good, h_t is the stock of durables at the end of period t, with $\gamma > 0$, $\gamma_h > 0$ being coefficients of relative risk aversion, and $\nu > 0$ being a utility weight. $\mathbb{E}_t [.]$ denotes the rational expectations operator conditional on the period-t information set. Households borrow internationally at a fixed gross real interest rate of $R > 1$. We assume that households are less patient than their foreign counterparts. Hence, the discount factor $0 < \beta < 1$ satisfies $\beta < R^{-1}$.

The flow budget constraint is:

$$c_t + q_t (h_t - h_{t-1}) - d_t = y f(e_t) - R d_{t-1}, \qquad t = 1, 2, ..., \infty, \tag{2}$$

where q_t is the price of durables (relative to that of nondurables), d_{t-1} is one-period debt carried over from last period, y is time-invariant income, and f is a function of a log-normally distributed income shock, e_t. We assume $f(e_t) \equiv \exp\left(-\frac{1}{2}\sigma_e^2\right)\exp(e_t)$, where σ_e^2 is the unconditional variance of e_t, and where the first term in f cancels the positive average level effect on income that log-normality introduces. We assume that e_t is driven by an AR(1) process:

$$e_{t+1} = \rho_e e_t + u_{t+1}^e, \qquad 0 < \rho_e < 1, \qquad u_{t+1}^e \sim \mathrm{N}\left(0, \sigma_e^2\right). \tag{3}$$

Despite free capital mobility, households may be constrained in their amount of borrowing. We assume that debt must be partly collateralized by durables. The contract stipulates that new borrowing, including interest, cannot exceed a time-varying fraction $s + s_t$ of the total expected value of durables:[3]

$$d_t \leq (s + s_t) \frac{\mathbb{E}_t [q_{t+1}] h_t}{R}, \qquad t = 1, 2, ..., \infty, \tag{4}$$

where s is the average loan-to-value (LTV) ratio, and s_t captures a stochastic part of the LTV with unconditional variance σ_s^2. It can be shown that (4) will be binding in the steady state, due to the assumption $\beta < 1/R$; see Appendix A. This implies a determinate steady state. The feature is shared by a multitude of papers involving economies characterized by credit frictions, as well as within small-open economy applications on 'sudden stops'; see, e.g., Kiyotaki and Moore (1997), Iacoviello (2005), Bianchi (2011), Eggertsson and Krugman (2012), Liu *et al.* (2013), Liu and Wang (2014), Justiniano *et al.* (2015), Jeanne and Korinek (2019), Schmitt-Grohé and Uribe (2021a), *inter alia*.[4]

The LTV shock evolves according to:

$$s_{t+1} = \rho_s s_t + u_{t+1}^s, \qquad 0 < \rho_s < 1, \qquad u_{t+1}^s \sim \mathrm{N}\left(0, \sigma_s^2\right), \tag{5}$$

and following a large literature we interpret variations in s_t as shorthand for stochastic

[3]Later in the analysis, we will also contemplate q_t in place of $\mathbb{E}_t [q_{t+1}]$ in the collateral constraint.

[4]A body of research on 'sudden stops' follows Mendoza (2010), where the credit constraint does *not* bind in the steady state. As we show in Section 6.4, our main findings are confirmed in an environment with an unconstrained steady state.

changes in the economy's financial conditions; see, e.g., Jermann and Quadrini (2012), Liu *et al.* (2013), Boz and Mendoza (2014), Bianchi and Mendoza (2018), and Jones *et al.* (2022).

In the DE, households maximize U subject to (2) and (4), taking as given $q_t > 0$ and the values of the states d_{t-1}, $h_{t-1} > 0$, e_t, and s_t.[5]

The optimality conditions in the DE are:

$$c_t^{-\gamma} = \Lambda_t, \tag{6}$$

$$\Lambda_t = \beta R \mathbb{E}_t \left[\Lambda_{t+1}\right] + \mu_t, \tag{7}$$

$$\Lambda_t q_t = \nu h_t^{-\gamma_h} + \beta \mathbb{E}_t \left[\Lambda_{t+1} q_{t+1}\right] + (s + s_t) \frac{\mathbb{E}_t \left[q_{t+1}\right]}{R} \mu_t, \tag{8}$$

where $\Lambda_t > 0$ and $\mu_t \geq 0$ are the multipliers associated with (2) and (4), respectively. We combine (6), (7), and (8) into the conventional Euler equations for optimal intertemporal consumption of perishable and durable goods, respectively:

$$c_t^{-\gamma} = \beta R \mathbb{E}_t \left[c_{t+1}^{-\gamma}\right] + \mu_t, \tag{9}$$

$$c_t^{-\gamma} q_t = \nu h_t^{-\gamma_h} + \beta \mathbb{E}_t \left[c_{t+1}^{-\gamma} q_{t+1}\right] + (s + s_t) \frac{\mathbb{E}_t \left[q_{t+1}\right]}{R} \mu_t. \tag{10}$$

3 Equilibrium and solution procedure

The market for durables is simplified by assuming that supply is constant:

$$h_t = h > 0, \qquad t = 0, 1, 2, \ldots, \infty, \tag{11}$$

holds in all periods. Based on this, we can then state:

Definition 1. *The decentralized equilibrium is a set of functions $d(\cdot)$, $c(\cdot)$, $q(\cdot)$, and $\mu(\cdot)$ that,*

[5]Further on, we will also consider the welfare properties of the CEE, so as to assess the role of pecuniary externalities.

conditional on d_{t-1} and $z_t \equiv [e_t, s_t]$, satisfy (2), (4), (9), (10). Therefore, this equilibrium satisfies:

$$c\left(d_{t-1}, z_t\right) + Rd_{t-1} = yf\left(e_t\right) + d\left(d_{t-1}, z_t\right), \tag{12}$$

$$c\left(d_{t-1}, z_t\right)^{-\gamma} = \beta R \mathbb{E}_t\left[c\left(d\left(d_{t-1}, z_t\right), z_{t+1}\right)^{-\gamma}\right] + \mu\left(d_{t-1}, z_t\right), \tag{13}$$

$$\begin{aligned} c\left(d_{t-1}, z_t\right)^{-\gamma} q\left(d_{t-1}, z_t\right) = {} & \nu h^{-\gamma_h} + \beta \mathbb{E}_t\left[c\left(d\left(d_{t-1}, z_t\right), z_{t+1}\right)^{-\gamma} q\left(d\left(d_{t-1}, z_t\right), z_{t+1}\right)\right] \\ & + \left(s + s_t\right) \frac{\mathbb{E}_t\left[q\left(d\left(d_{t-1}, z_t\right), z_{t+1}\right)\right]}{R} \mu\left(d_{t-1}, z_t\right), \end{aligned} \tag{14}$$

$$0 = \mu\left(d_{t-1}, z_t\right)\left[d\left(d_{t-1}, z_t\right) - \left(s + s_t\right) \frac{\mathbb{E}_t\left[q\left(d\left(d_{t-1}, z_t\right), z_{t+1}\right)\right] h}{R}\right], \tag{15}$$

where (15) is the complementary slackness condition associated with (4) and $\mu\left(d_{t-1}, z_t\right) \geq 0$, and where the exogenous disturbances, z_t, evolve according to (3) and (5).

Note that the exogenous stochastic variables, e_t and s_t, enter the equilibrium conditions (12)–(15) so that, when considering different mean-preserving spreads of each shock (Rothschild and Stiglitz, 1970, 1971), we do not introduce arbitrary exogenous mean level effects.[6] Exogenous level effects and their accompanying biases have long been acknowledged in the literature on uncertainty shocks in business cycles; see, e.g., Rankin (1994).

We solve the non-linear system (12)–(15) numerically. We follow Jeanne and Korinek (2019), adapting their approach to models with collateral constraints based on the expected-future collateral price. The state space spanned by d_{t-1} and z_t is discretized by 2,501 points for debt and a five-state Markov chain for each of the shocks. Through Euler-equation iteration, we obtain approximate policy functions $d_t = d\left(d_{t-1}, z_t\right)$, $c_t = c\left(d_{t-1}, z_t\right)$, $q_t = q\left(d_{t-1}, z_t\right)$, and $\mu_t = \mu\left(d_{t-1}, z_t\right)$. The recursive nature of the policy functions enables us to solve for the value function, $V_t \equiv V\left(d_{t-1}, z_t\right) = \left[1/(1-\gamma)\right] c\left(d_{t-1}, z_t\right)^{1-\gamma} + \left[1/\left(1-\gamma^h\right)\right] h^{1-\gamma_h} + \beta \mathbb{E}_t\left[V\left(d\left(d_{t-1}, z_t\right), z_{t+1}\right)\right]$, which will be the basis for welfare analyses. The solution algorithm is described in detail in Appendix B.

4 Calibration

Following Bianchi and Mendoza (2018), we calibrate the model using OECD country data, while resorting to U.S. data in a few cases where data is not available for all OECD members. One period is interpreted as a quarter, such that $R = 1.01$ implies the commonly assumed 4% annual real interest rate, as is standard in small open economy models (e.g., Bianchi, 2011; or Schmitt-Grohé and Uribe, 2021a). We calibrate the value of

[6]We confirm this property by a series of Monte Carlo experiments (results available upon request).

β such that—given the characteristics of the shocks, which we describe below—the model matches the average skewness of consumption growth across all OECD countries for the period 1980:Q1 to 2019:Q4, which is -0.9.[7] As discussed by Jordà *et al.* (2020), the negative skewness of consumption in the data plays a crucial role in the evaluation of the welfare effects of business cycles, and is, therefore, important to match. In our model, the value of β is a key determinant of the frequency with which the collateral constraint becomes nonbinding, and therefore—this being the only source of asymmetry in the model—of the degree of skewness in consumption. The calibration results in a value of $\beta = 0.967$; a number closely in line with those used in many existing studies (e.g., Benigno *et al.*, 2013; Reyes-Heroles and Tenorio, 2020; and Jensen *et al.*, 2020). The average LTV ratio, s, is set to 0.8, which is in line with the cross-country evidence reported by Calza *et al.* (2013) for a range of advanced economies, and with the values used in Bianchi and Mendoza (2018). Households' coefficients of relative risk aversion, γ and γ_h, are set to 2, in line with microeconometric evidence (e.g., Attanasio and Weber, 1995) and much of the existing literature (e.g., De Santis, 2007; Benigno *et al.*, 2013; and Sosa-Padilla, 2018). Both the steady-state income and the stock of durables are normalized to 1. The price of durables is then determined by the preference parameter, ν. We calibrate it to obtain an annualized ratio of household debt to output of 0.63, which is the median value for this ratio across the group of OECD countries; see IMF (2017). This implies setting $\nu = 0.048$.

The parameters related to the shock processes are calibrated as follows. The income shock is parameterized so that the income process in the model matches the average standard deviation and autocorrelation of the gross domestic product at business-cycle frequencies across all OECD countries for the period 1980:Q1 to 2019:Q4.[8] This yields $\sigma_e = 0.015$ and $\rho_e = 0.908$. As for the financial shock, we are not aware of quarterly data for assets and liabilities of the household sector across OECD countries for a sufficiently long period.[9] Instead, we focus on U.S. data and use the series for the LTV ratio of households constructed by Jensen *et al.* (2020) based on Flow of Funds data. We then set the parameters of the financial shock process so that the movements in the LTV ratio in the model match fluctuations in its empirical counterpart at business-cycle frequencies

[7] We obtain data from the OECD World Economic Outlook database. For some OECD countries, the available data sample is shorter, and we simply include all available quarters for each country.

[8] To focus on fluctuations at business-cycle frequencies, we apply a band-pass filter with bounds of 6 and 32 quarters, as is common in the literature.

[9] The OECD collects data for most member countries only since the late 1990s or even later. This dataset is, therefore, heavily influenced by the Global Financial Crisis of 2007-09 and would thus lead to larger financial shocks. As will become clear below, this would only strengthen our main findings.

Parameter	Description	Value
R	Gross real rate of interest	1.01
β	Discount factor	0.967
γ	CRRA, perishable consumption utility	2
γ_h	CRRA, durable consumption utility	2
ν	Utility weight, durable consumption	0.048
s	Average LTV ratio	0.8
y	Average income	1
h	Supply of durables	1
σ_e	Standard deviation of the income shock	0.015
ρ_e	Autoregressive parameter of the income shock	0.908
σ_s	Standard deviation of the financial shock	0.016
ρ_s	Autoregressive parameter of the financial shock	0.934

Table 1: Baseline parameter values.

over the period 1980:Q1-2019:Q4. This requires setting $\sigma_s = 0.016$ and $\rho_s = 0.934$. All parameter values are summarized in Table 1.

5 The welfare effects of business-cycle fluctuations

To measure the welfare cost of business cycles, we follow Lucas (1987), and ask by what percentage the stochastic consumption path should be increased to obtain the same unconditional welfare as in the same economy with no shocks. As shown in Appendix C, this number is given by:

$$\lambda = 100 \left[\left(\frac{\mathbb{E}\left[\overline{V}\left(d_{t-1}\right)\right] - u^h}{\mathbb{E}\left[V\left(d_{t-1}, z_t\right)\right] - u^h} \right)^{\frac{1}{1-\gamma}} - 1 \right], \tag{16}$$

where $\overline{V}\left(d_{t-1}\right)$ denotes equilibrium welfare in an economy with no shocks, and where $u^h \equiv [1/(1-\beta)]\left[\nu/(1-\gamma_h)\right] h^{1-\gamma_h}$.[10] Based on this metric, the unconditional cost of business cycles amounts to -0.2128% of quarterly consumption in perpetuity, i.e., a net welfare gain. While not being large in absolute value, this is more than one order of magnitude larger than Lucas's (1987) original number, and it is the existence of a gain that is of most interest in our analysis.

[10] Appendix C also reports the analytical details about alternative welfare computations in this section.

Conditional welfare We find it insightful to condition the welfare measure on both the stock of debt and the realizations of the shocks. To this end, the following measure of *conditional* welfare loss of business cycles can be derived:

$$\lambda^c\left(d_{t-1}, z_t\right) = 100 \left[\left(\frac{\overline{V}\left(d_{t-1}\right) - u^h}{\mathbb{E}_t V\left(d_{t-1}, z_t\right) - u^h}\right)^{\frac{1}{1-\gamma}} - 1\right]. \tag{17}$$

The left panel of Figure 1 shows that, irrespective of the history of debt, when shocks take on their average values, then $\lambda^c < 0$. Hence, the presence of business cycles is welfare-enhancing, particularly when initial debt is close to the deterministic steady state and, therefore, the economy is prone to switching to a regime in which the constraint does not bind.

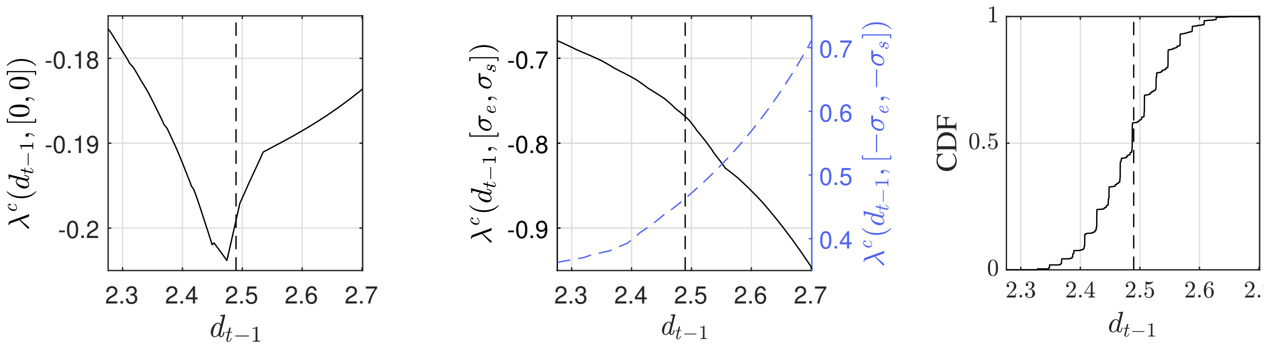

Figure 1: Conditional welfare losses and the stationary debt distribution. Left panel: Both shocks are initially at their means. Center panel: Both shocks are initially one s.d. higher (left axis, solid line), or one s.d. lower (right axis, dashed blue line) than their means. Right panel: Stationary cumulative distribution of debt. In each panel, the dashed vertical line denotes the deterministic steady-state debt level.

Now, consider the central panel of Figure 1. Here we examine two opposite initial conditions. A 'bad' state, where both exogenous shocks are one standard deviation below their means, and a 'good' state, where both shocks are one standard deviation above. Notice that, irrespective of initial debt, there is a cost of business cycles, conditional on a bad economic state (right axis). In fact, the magnitude of the cost can be conspicuous for an economy in high debt (0.5–0.7% of consumption). In the good state, instead, the opposite holds true (left axis): Irrespective of initial debt, we appreciate a business cycle gain, which increases over the support of d_{t-1} (0.7–0.9% of consumption).

In light of this asymmetry—and to take a first step towards uncovering the origins of the welfare gain of business fluctuations—it is important to quantify the chances that debt is endogenously driven to a 'costly' region of its support. In this respect, the right panel of Figure 1 reports the stationary cumulative distribution function of debt. The vertical line corresponds to mean debt in the deterministic economy, which amounts to 2.4951.[11] Two insights are offered by this exercise: First, the distribution of debt is rather narrow around the deterministic level; second, the distribution is skewed to the left, so that 58% of the time d_t is lower than its counterpart in the deterministic case. Hence, it is relatively rare that debt may actually end up in the region where business cycles are very costly.

Ergodicity Prior to examining the factors behind the emergence of a welfare gain from business fluctuations, we should acknowledge that our approach entails a comparison between welfare in the presence of an occasionally binding collateral constraint and the corresponding non-stochastic steady state, where the constraint binds invariably. While this aspect may bear some quantitative relevance for the emergence of gains from business fluctuations, our subsequent analysis emphasizes that $\lambda < 0$ is by no means a property that is hardwired in the model economy we consider. Nevertheless, at this stage of the analysis, we find it important to test the robustness of our result with respect to the choice of the benchmark economy to which we compare the one under uncertainty. Specifically, we consider *ergodicity* as an alternative notion of steady state, and compute the average of the value function in the deterministic setting by employing the density of debt from the stochastic model (see Appendix C for further analytical details). The resulting λ amounts to -0.2063%; that is, a gain of a size similar to that of the non-stochastic steady-state benchmark.

6 Dissecting the gain

This section presents various analyses and exercises aimed at rationalizing our findings. We begin by elucidating the source of the welfare gain in the presence of uncertainty. Next, we isolate and examine the role of each of the two shocks separately to understand their specific qualitative and quantitative contributions to the emergence of welfare gains. We then turn to an exploration of the significance of pecuniary externalities, which play a crucial role in debt determination and welfare in economies where the borrowing limit is influenced by collateral prices. For this purpose, we compare welfare in the DE with that

[11] In the stochastic economy, instead, the mean is 2.4899; i.e., a slightly lower figure arises due to precautionary saving. This difference also explains why λ^c reaches a trough slightly to the left of the deterministic steady-state debt level in the left panel of Figure 1.

in the CEE, both in our baseline economy and in an alternative setting, where the current price of the collateral asset is used in place of the expected-future price. Finally, we present some robustness exercises to analyze the sensitivity of λ to certain fundamental properties of the model specification and parameter values.

6.1 How can business cycles be beneficial?

A necessary premise for our analysis is to recall that, given the concavity of the utility function, households would prefer a stable consumption stream over one that fluctuates around the same average, assuming all other factors remain constant. Taken in isolation, this fluctuations effect would result in a welfare cost of business cycles, although a quantitatively small one.

Our key insight is that, in a collateral-constrained environment where the economy alternates between a binding constraint and a frictionless state, a welfare gain may emerge when households make use of uncertainty to their own advantage. To this end, the kink in debt determination is crucial as, in conjunction with prudence, it shapes convexity in households' marginal utility of consumption, thus affecting the extent of precautionary saving and the frequency of episodes in which households find themselves financially unconstrained.[12] This trait clearly emerges in households' consumption policy function, where the negative relationship between consumption and the initial stock of debt becomes even steeper beyond the kink (see the top-right panel of Figure E.2 in Appendix E.4).

The presence of a collateral constraint induces households to self-insure, as they foresee the possibility that current and future constraints may bind (in this regard, see also Carroll *et al.*, 2021). This can be seen by iterating their nondurable consumption Euler equation (9) forward:

$$c_t^{-\gamma} = \mu_t + \beta R \mathbb{E}_t \left[\mu_{t+1} + \beta R \mu_{t+2} + (\beta R)^2 \mu_{t+3} + ... \right]. \tag{18}$$

In a stochastic environment, the tightness of the financial constraint varies: Therefore, at any point in time, even if $\mu_t = 0$, the expected term on the right side of (18) must be non-negative, in light of $\mu_t \geq 0$, $\forall t$. Since households fear getting consecutive bad income realizations that would push them toward the borrowing limit, thus forcing them into painful deleveraging, they reduce leverage for precautionary reasons, as uncertainty

[12]The role of prudence will be examined in isolation in Section 6.4.

increases. This aspect is of central importance in rationalizing the connection between welfare and uncertainty in the present economy.

To substantiate this claim, we turn to Figure 2, which reports the results from an exercise in which we vary the standard deviations of the two shocks that perturb the model economy. That is, we multiply the vector of standard deviations $[\sigma_e, \sigma_s]$ by a constant, k, and then report some key model statistics as k gradually increases from zero to five, with $k = 1$ representing the outcome at the baseline calibration. The left panel displays the welfare cost of business cycles (left axis) against the frequency with which the borrowing constraint becomes nonbinding (right axis). Before diving into the key highlights of this exercise, a word of caution is warranted: *In theory*, even when shocks are very small, agents face a non-zero probability of becoming unconstrained, and therefore engage in precautionary saving in an attempt to reduce the frequency at which the credit constraint binds. *In practice*, though, the model is solved by discretizing the state space, including the shock processes, effectively truncating the distributions from which shocks are drawn. Therefore, agents only appear to attach a non-zero probability to 'nonbinding episodes' when the standard deviation of the shocks is sufficiently high. This coincides with the point at which instances of a nonbinding constraint are actually observed, as reflected by the dashed green line in the left panel of Figure 2. Thus, in the remainder of the analysis, we will consider the frequency of nonbinding constraints *ex post* as an indicator of the probability of becoming unconstrained *ex ante*.

According to the left panel of Figure 2, when uncertainty is relatively low, its influence on the precautionary-saving attitude of households stemming from endogenous switching is negligible. In fact, the economy fluctuates within a neighborhood of the steady state, where the constraint is binding. In these circumstances, the model unambiguously entails that business cycles are costly, due to the fluctuations effect. As k is gradually raised and shocks hitting the economy reach a certain magnitude, the endogenous switching effect quickly becomes dominant, leading to the emergence of welfare gains from the business cycles. This happens as the frequency at which the borrowing constraint does not bind starts to increase, both because of the direct impact of progressively larger shocks, and because the ensuing strengthening of households' precautionary motive increasingly allows them to avoid hitting against the financial constraint. In other words, households may neglect to fully utilize expansions in their borrowing capacity that arise from positive shocks, thus adhering to their inherent consumption-smoothing behavior. In fact, they might continue to remain unconstrained even in the face of negative shocks, although the constraint will eventually become binding again over time.[13]

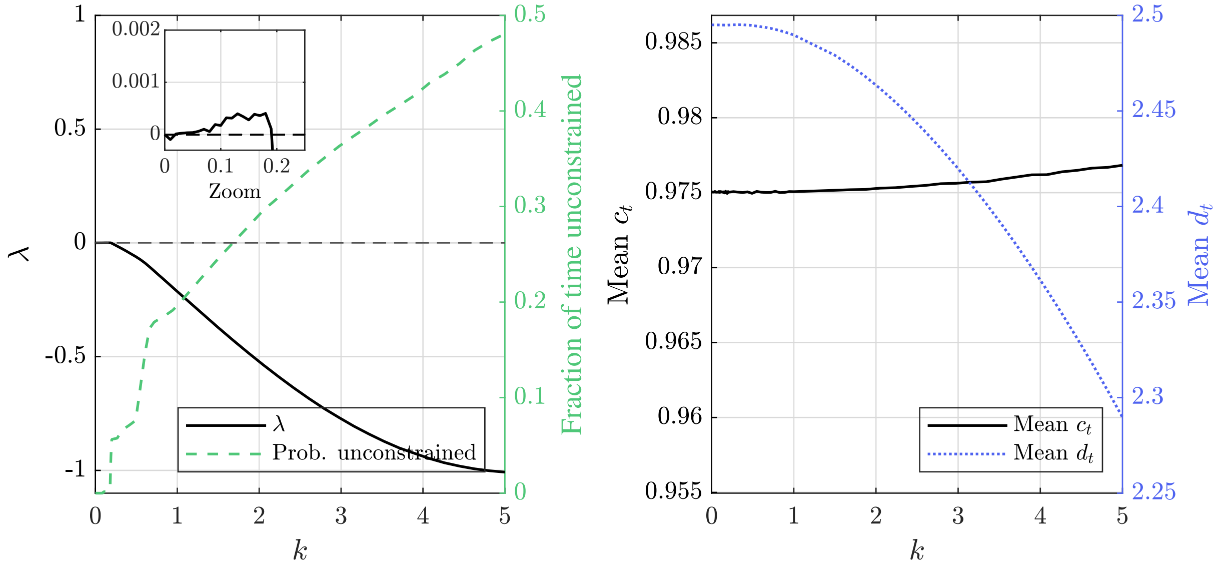

Figure 2: Welfare cost of business cycles and the frequency of episodes in which the collateral constraint is slack (left panel) and the pattern of average consumption and debt (right panel) with respect to scaling σ_s and σ_e by a common factor k. All the other parameters are at their baseline values.

While the left panel of Figure 2 allows us to establish a tight connection between uncertainty, the frequency of episodes in which the borrowing constraint does not bind, and the emergence of welfare gains, the right panel of the figure sheds light on the underlying source of the gain. Here, we report average consumption (solid black line, left axis) against the average stock of debt (dotted blue line, right axis). The pattern of average debt reflects households' precautionary behavior, as discussed above: As shocks become larger, households increase their precautionary savings, thus reducing their indebtedness. In turn, lower debt allows them to enjoy higher average consumption—as predicted by their consumption policy function—thus paving the way for the emergence of a welfare gain.

To emphasize the importance of precautionary saving as a way for households to take advantage of uncertainty and obtain welfare gains, we also run the model with perfect foresight. In this case, removing uncertainty implies that households' saving attitude is solely driven by a combination of intertemporal and smoothing motives, as captured by their discount factor and the curvature of their utility function. In this case, we observe

[13]We provide an example of this based on a sample of simulated data in Figure D.1 in Appendix D.

a welfare cost of business cycles, with $\lambda = 0.028\%$.[14] This indicates that, not being faced with the need/option to save as a precautionary expedient to reduce the probability of hitting their borrowing limit, households incur a cost from living in the stochastic economy, where the fluctuations effect dominates.

6.2 On the role of different shocks

The next step in the analysis involves examining the specific role of the two shocks in the model, as this will facilitate a deeper understanding of the mechanics leading to gains/losses from uncertainty. To this end, Figure 3 reports λ conditional on switching off either of the shocks at a time, while varying the standard deviation of the shock at play. Consider the case of no financial shocks (i.e., $\sigma_s = 0$ in the left panel of the figure): For an initial narrow range of the standard deviation of the income shocks hitting the economy, households realize that episodes of nonbinding constraints are unlikely to occur. In fact, when the frequency of such episodes remains at zero, we observe that $\lambda > 0$ (albeit marginally). Raising σ_e further allows endogenous switching to gain traction, leading to a welfare gain for a rather wide spectrum of values of the standard deviation of the income shock. However, as income fluctuations become conspicuous, λ is driven back into the 'costly' region, as negative realizations of the shock become very painful. On the contrary, in the absence of income shocks ($\sigma_e = 0$), raising σ_s (as shown in the right panel of Figure 3) results in an increasing gain from business fluctuations, starting from the point where a positive frequency of nonbinding episodes is observed.[15]

The different role played by the two shocks is reminiscent of the result of Cho *et al.* (2015), who show that fluctuating economies may enjoy higher welfare, as compared with their no-shock counterparts, but only in the presence of multiplicative shocks. A positive *mean effect* of uncertainty is at work in their case—along with the fluctuations effect—as compared with situations in which shocks enter additively: Multiplicative shocks have the potential to increase mean output and / or consumption, allowing consumers to take advantage of uncertainty by working harder and investing more during expansionary periods. On the contrary, when uncertainty enters the economy additively, it has no beneficial effect on the choices that can be adjusted to it. In the present setting, endogenous switching exerts a positive impact on the average level of consumption, similar to the

[14]Each perfect foresight simulation lasts 400 periods. The results are computed from 2,500 simulations.

[15]Note that the pattern observed in the right panel of Figure 2 is confirmed conditional on each of the two shocks: As shown in Figure D.2 in Appendix D, increasing the standard deviation of either shock—with the other shock switched off—leads to higher average consumption and lower average debt.

mean effect described by Cho *et al.* (2015). However, the two effects are different in nature: Endogenous switching can be triggered not only by LTV shocks—which enter multiplicatively into the Euler equation for durables, (10)—but also by income shocks, which enter in a purely additive manner into the budget constraint, (2). This can be seen from the left panel of Figure 3, which reports a welfare gain over a wide region of the σ_e-support, conditional on financial shocks being turned off. In other words, households can exploit uncertainty arising from either type of shock—be it additive or multiplicative—in our setting.

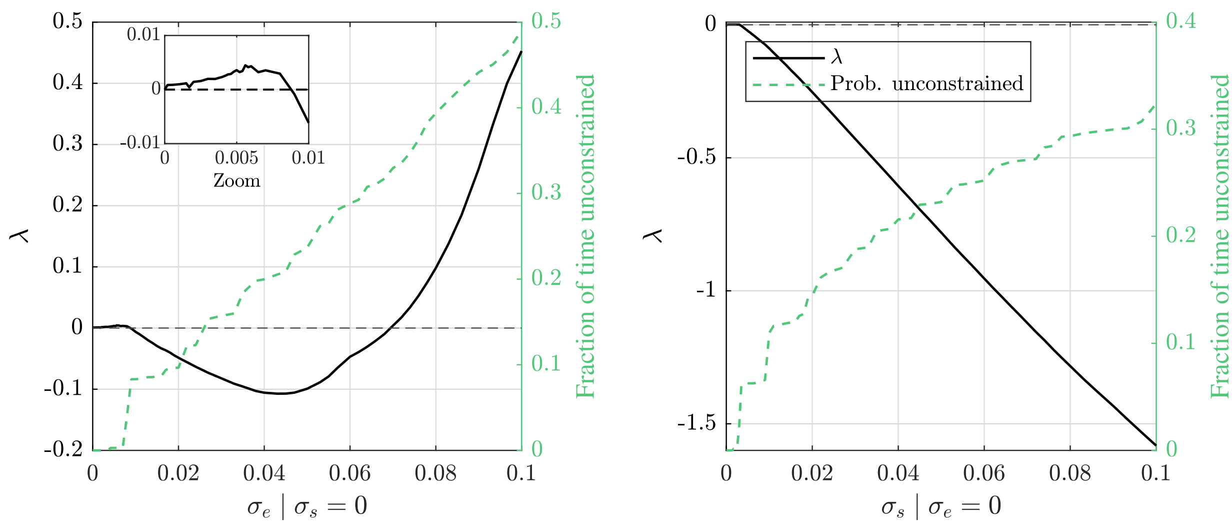

Figure 3: Varying uncertainty in the baseline economy. In each panel, the continuous (black) line reports λ for different standard deviations of a given shock, conditional on the other shock being switched off. The dashed (green) line indicates the frequency of episodes in which the financial constraint is slack. All the other parameters are at their baseline values.

6.3 What role for pecuniary externalities?

In the present environment, an important question is whether households' optimizing behavior shapes the gain from business fluctuations by affecting the equilibrium price of the collateral asset through inefficient borrowing (see, e.g., Lorenzoni, 2008 and Bianchi, 2011). Pecuniary externalities may, in fact, prove crucial in that they generate an uninternalized welfare cost, whereby an increase in borrowing today tightens households' ability to borrow tomorrow. To address this point, envisaging a social planner who internalizes the price effect of borrowing in the collateral constraint—thus attaining the CEE—should be informative about the potential role of the pecuniary externality in generating welfare gains from uncertainty. Specifically, we are interested in: *i*) comparing welfare under uncertainty in the CEE with the steady-state benchmark; *ii*) comparing welfare under uncertainty in the DE with that in the CEE. According to this strategy, comparison *i*) should inform us on the effects of uncertainty on welfare, in an environment without pecuniary externalities. Comparison *ii*), instead, is informative about the role of pecuniary externalities in an environment with uncertainty. The logic from previous research is that welfare in the CEE should exceed welfare under the DE (see, e.g., Bianchi, 2011).

At this stage of the analysis, it is important to stress the emergence of a distinctive equivalence between the CEE and the DE of our economy—in the vein of Ottonello *et al.* (2022)—when the collateral constraint (4) features the expected-future price of the collateral asset, $\mathbb{E}_t q_{t+1}$, on its right-hand side. Proposition 1 formalizes this result.

Proposition 1. *The decentralized equilibrium of the baseline economy is constrained-efficient.*

Proof. See Appendix E.

As extensively discussed by Ottonello *et al.* (2022)—as well as in Appendix E, for our framework—the shadow value of borrowing in the DE is a rescaled version of that in the CEE, implying that no macroprudential policy is desirable in our environment. To confirm this result, the first row of Figure 4 reports λ in the CEE, conditional on each shock at a time and for varying standard deviations, while the second row compares welfare in the DE *vis-à-vis* that in the CEE: It is immediate to notice that the equivalence result of Proposition 1 is confirmed for any standard deviation of the underlying shocks. Such a defining property of models featuring the expected-future price of the collateral asset in (4) makes comparisons *i*) and *ii*) trivial, *de facto*, at this point of our analysis.

A key insight from Ottonello *et al.* (2022) is that the DE-CEE equivalence breaks down when the collateral constraint is written in terms of the *contemporaneous* asset price, q_t, instead of the *expected-future* price, $\mathbb{E}_t q_{t+1}$, implying that there is a role for macroprudential policies in this case. Motivated by their findings, we also consider an alternative to our model that features the contemporaneous asset price in the collateral constraint. We confirm the breakdown of the DE-CEE equivalence in this setting, as Appendix F discusses in detail. Nevertheless, we also record a welfare gain from business cycles in this environment, both in the DE and in the CEE: At the baseline calibration, λ amounts to -0.0162% and -0.0464%, respectively. This suggests that the pecuniary externality does not affect the very emergence of a welfare gain, but only its size. In fact, by internalizing the externality in the CEE, the social planner attains higher welfare, as expected on *a priori* grounds.[16]

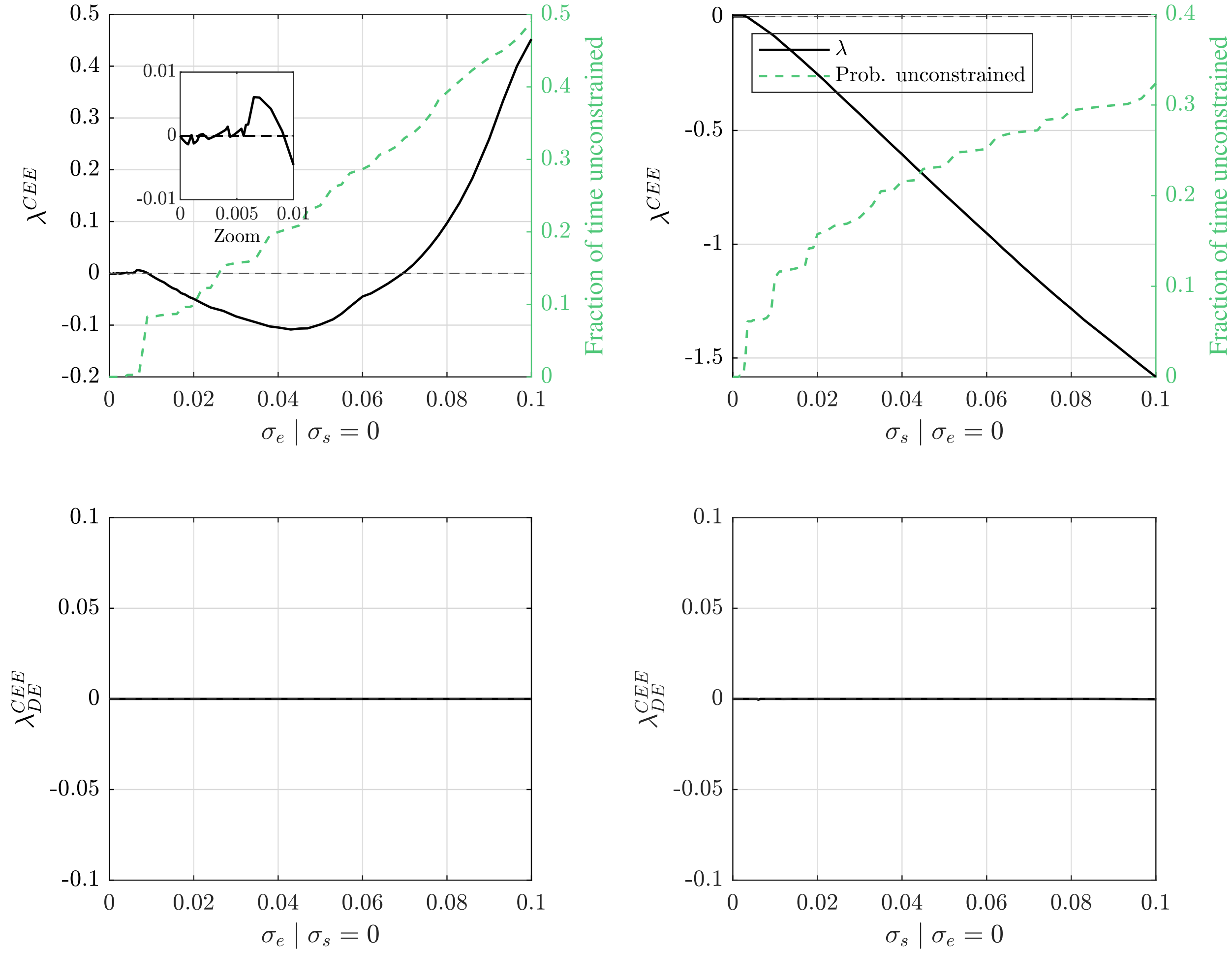

Figure 4: Varying uncertainty and welfare. In each panel, the continuous (black) line reports a welfare measure/comparison for different standard deviations of a given shock, conditional on the other shock being switched off (all the other parameters are at their baseline values). Specifically, row 1 computes λ in the CEE, while row 2 compares the value function in the DE with that in the CEE. The dashed (green) line in row 1 indicates the frequency of episodes in which the financial constraint is slack.

We report additional results from the model featuring q_t as the collateral price in Appendix F. In this respect, it turns out—as stressed by Ottonello *et al.* (2022) and Juul (2023)—that assuming collateral to be valued at its current price is not innocuous, for the price itself is a function of current debt. As a result, the level of debt at any point in time becomes a (major) determinant of borrowers' ability to take on more debt and, thus, to

[16]Beyond a certain threshold for σ_s, however, collateral shocks prove key in producing higher welfare in the DE, as compared with the CEE. For details, see Figure F.3 and the discussion in Appendix F.4.

deleverage when necessary. This opens up to the emergence of endogenous debt cycles, in which increasing levels of debt tighten the borrowing constraint, and eventually lead to unstable dynamics; see Schmitt-Grohé and Uribe (2021b). Such possibility is ruled out when the expected-future price of the collateral asset determines borrowers' ability to take on more debt, as q_{t+1} does not depend on the current state of debt, in equilibrium; see Juul (2023).[17]

6.4 Robustness

As a final step in our analysis, we now seek to explore the sensitivity of our main findings with respect to some alternative model settings and parameter values.

6.4.1 Alternative model settings

As shown in Section 6.3, the way collateral is priced in the financial constraint assumes a certain importance, although mainly from a quantitative viewpoint. In light of this, we first consider two exercises aimed at testing the emergence of welfare gains under alternative assumptions about endogenous movements in the relative price of durables. Next, we assess the role played by our assumption that the collateral constraint is binding in the steady state of the model.

Fixed credit limits We first consider the special case of a fully exogenous credit limit, where the collateral is always pledged at the steady-state value, such that (4) is replaced by $d_t \leq (s + s_t)\frac{qh}{R}$. In other words, we do allow for endogenous movements in the relative price of durables, but switch off any influence this may have through the collateral value. In this case, we obtain $\lambda = -0.016\%$ at our baseline calibration, indicating that there is still a welfare gain, yet of an order of magnitude lower than in our baseline model.[18] This highlights the quantitative importance of the financial accelerator effect of Kiyotaki and Moore (1997): In the absence of endogenous movements in the collateral value, endogenous switching is inevitably dampened. Notably, this exercise also corroborates our message that pecuniary externalities are not the driver of our main finding. With a fixed credit limit, there is no pecuniary externality at play, and the problem of the social planner ensuring the CEE, therefore, coincides with the one faced by households in the DE.

[17] This aspect becomes evident when comparing the policy functions of d_t—both in the DE and the CEE—from the model with the expected-future collateral price (Figure E.2 in Appendix E.4) with those from the current-price configuration (Figures F.1 and F.2 in Appendix F.4, respectively).

[18] In Appendix G, we examine how this number changes with the magnitude of each of the two shocks.

Partial vs. general equilibrium The former exercise shows how endogenous movements in the collateral asset can be key in generating a significant welfare gain. We now switch off any movement in the relative price of durables and abstain from imposing the corresponding market clearing condition, (11), thus devising a partial equilibrium economy (see Appendix H for further details). This twist leads to a substantial loss of welfare in the presence of uncertainty ($\lambda = 6.0515\%$ for our baseline calibration and when setting the asset price at its steady-state level from our baseline model). This is primarily due to a structural trait of the model in partial equilibrium, as the stock of durables now represents an endogenous state variable (along with debt) that exerts a strong grip on debt determination, and imposes particularly burdensome deleveraging in response to adverse shocks, especially when the asset price is relatively high and/or conditional on credit-limit shocks. In fact, it is possible to show that, when setting $\sigma_e = 0$, raising σ_s produces large losses from business fluctuations (see the right panel of Figure H.2 in Appendix H). Having a direct effect on debt determination, negative credit-limit shocks impose large reductions in both the stock of durables and nondurable consumption, so as to deleverage and satisfy the collateral constraint.[19] Such a feature does not characterize the general equilibrium setting, where the variable determining endogenous movements in the collateral value is the asset price, which makes the resulting deleveraging episodes smoother, even when we consider a current-price version of the financial constraint. Nonetheless, it is important to stress that Figure H.2 also shows that we may still observe increasing welfare gains from business cycles when raising the volatility of income shocks alone, conditional on $\sigma_s = 0$ (left panel). Once more, this confirms that endogenous switching is key to observing welfare gains, even in a setting where endogenous movements in the relative price—as well as exogenous shocks to credit limits—are completely shut off.

Unconstrained steady state In the baseline model considered so far, we have made the assumption that the collateral constraint (4) is binding in the deterministic steady state of the economy. Another stream of work—building on Mendoza (2010)—has instead entertained the assumption that the constraint is nonbinding in the steady state. To investigate the implications of this choice for the potential emergence of welfare gains, we solve a version of our model that features an unconstrained steady state. The details are provided in Appendix I. Under a calibration analogous to that described in Section 4, we confirm the existence of a welfare gain, although smaller than in our baseline economy, as we obtain

[19] See, in this respect, the comparison between the policy functions in partial and in general equilibrium, as reported by Figure H.1 in Appendix H.

$\lambda = -0.0432\%$. In essence, the driver of the welfare gain is the same as above: Uncertainty induces households to increase their precautionary savings in order to remain financially unconstrained as often as possible, leading to lower average debt, higher average consumption, and overall higher welfare (see Figure I.2 and I.3 in Appendix I).[20]

6.4.2 Comparative Statics

We now turn to some comparative-statics exercises aimed at examining the relevance of some key parameters in the model.

Discount factor It is important to start by observing the impact of households' degree of patience on unconditional welfare, as this has a tangible impact on their saving/consumption attitude. To this end, we examine welfare over a wide range of β's. The left panel of Figure 5 shows that, as consumers start at an implausibly high degree of impatience, the economy with uncertainty is welfare-dominated by the deterministic scenario. In these circumstances, the credit constraint binds tightly, such that shocks never lead to the occurrence of episodes where agents are unconstrained. In fact, the endogenous switching effect is insubstantial when consumers are very impatient, so the fluctuations effect prevails.

However, as β increases beyond a certain threshold—which lies well below the range of values typically considered in calibrations based on quarterly data—the cost of business cycles eventually translates into a steadily increasing gain. As households' *intertemporal saving motive* strengthens with their degree of patience, endogenous switching gains traction.

Risk aversion The tension between financial tightness and the intensity of households' precautionary saving motive is central to our story, as discussed above. We have already unveiled the role of self-insurance stemming from the kink in debt determination. Furthermore, self-insurance is intimately connected with prudence (Kimball, 1990), which is indexed by $1+\gamma$, in our setting. With this in mind, we examine how risk aversion impacts unconditional welfare in the central panel of Figure 5.

[20]Notably, the policy functions for consumption and debt also feature a kink in this model setting, as seen from Figure I.1 in Appendix I. The smaller gain obtained in this case primarily reflects the fact that, at the baseline calibration, credit-limit shocks are unlikely to trigger endogenous switching, so that only income shocks 'contribute' to the welfare gain, as we discuss in Appendix I.

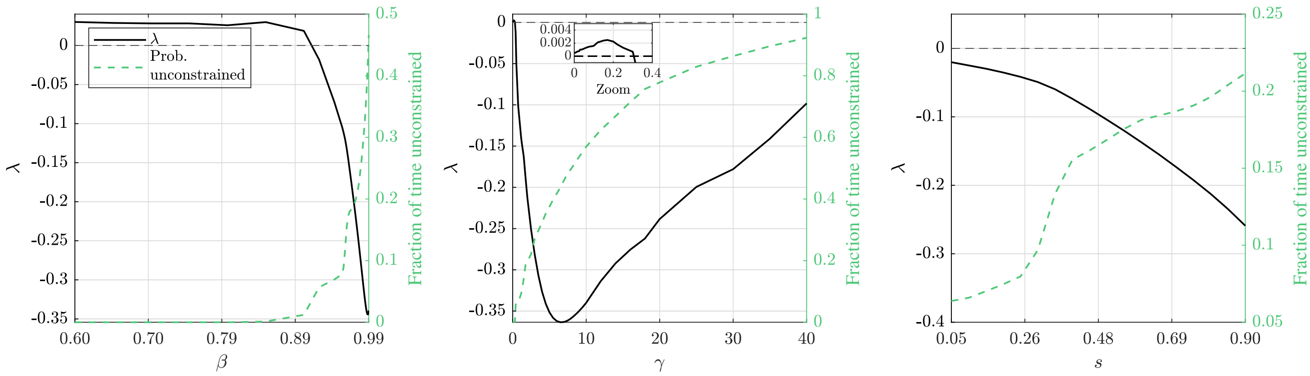

Figure 5: Welfare costs of business cycles for different values of the discount factor (left panel), the coefficient of relative risk aversion (center panel), and the steady-state LTV ratio (right panel). All the other parameters are at their baseline values.

Notably, when consumers are risk-neutral ($\gamma = 0$), the borrowing constraint binds and λ is virtually zero. When households become slightly risk-averse, fluctuations become costly, albeit very little (peaking at $\lambda \approx 0.002\%$ when $\gamma = 0.17$). As γ increases further, λ dives into the negative territory. Higher risk aversion reinforces households' aim at finding themselves in a financially-unconstrained regime more and more frequently, thus reducing leverage, and enjoying higher average consumption. However, as γ increases further, λ reverts its pattern (although it remains negative over a plausible range of values for households' relative risk aversion). This happens for two reasons: First, despite households becoming increasingly risk averse, thus reducing debt and allowing for a rise in average consumption, the economy is more prone to infrequent—yet large—drops in consumption. Second, as γ increases, the fluctuations effect becomes stronger.

LTV ratio As depicted in the right panel of Figure 5, business fluctuations are beneficial even at very low LTV ratios. In fact, the gain increases as we move to the right over the s-support, similar to what we observe when the volatility of the *stochastic* part of the credit limit increases. All else equal, higher credit availability entails that fluctuations in the collateral value have a stronger impact on the shadow value of borrowing. In turn, changes in the latter are key in modulating households' precautionary saving behavior.

7 Concluding remarks

This paper quantifies the welfare cost of business cycles in a credit economy where households may or may not find themselves financially constrained. Welfare tends to be higher when the economy is subject to fluctuations, as compared with the uncertainty-free benchmark case. Households' precautionary motive to stay clear of the credit constraint is the driver of the gain, as it stimulates higher average consumption by reducing debt.

Although derived in a stylized framework, this basic yet clear insight may have crucial implications for stabilization policies that influence the frequency at which collateral constraints become slack. In light of this, future research should aim at implementing medium-scale models featuring a richer set of frictions and propagation channels, to trace out the implications of our argument for the design of macroprudential policies. In related work, Jensen *et al.* (2018) have established a macroeconomic volatility trade-off that arises from collateral constraints not binding at all points in time: On one hand, a reduction of credit limits may dampen the asset-price sensitivity of those borrowers who remain credit constrained before and after the intervention; on the other hand, lower credit limits increase the frequency at which credit constraints bind, thus augmenting borrowers' sensitivity to fluctuations in credit availability. In a similar fashion, though in an environment with idiosyncratic risk and incomplete markets, Lee *et al.* (2020) show that stricter regulation of leverage in the banking sector inhibits households' ability to smooth consumption in response to idiosyncratic risk. Thus, while this type of restriction might exert a stabilizing effect at the macroeconomic level, it does not necessarily stabilize at the microeconomic level, potentially resulting in substantial welfare costs. In sum, both contributions indicate the importance of assessing agents' positions with respect to kinks in their policy functions, much like we have done in this paper.

On a more general note, our findings—along with those just discussed—support the message from other recent contributions emphasizing that the devil is in the detail when it comes to designing macroprudential policies based on theoretical models featuring borrowing constraints. Ottonello *et al.* (2022) conclude that further empirical research is needed to distinguish current-price from future-price collateral constraints, whereas Drechsel and Kim (2022) show that while collateral constraints typically lead to overborrowing, earnings-based borrowing constraints—which they argue to be no less relevant, from an empirical perspective—entail underborrowing, as compared with the socially optimal level. Our results add to this agenda by stressing the quantitative importance of facing nonbinding constraints, as well as the role played by different types of shocks to the economy, for normative purposes.

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Appendices

A Deterministic steady state

In the absence of shocks, (13) implies:

$$\mu = (c)^{-\gamma}(1-\beta R) > 0, \tag{A.1}$$

where undated variables denote deterministic steady-state values, and where the inequality follows from $\beta < 1/R$. From (15), we therefore get:

$$d = s\frac{qh}{R}. \tag{A.2}$$

From (12) and (A.2) we obtain:

$$c + sqh\left(1 - R^{-1}\right) = y. \tag{A.3}$$

By virtue of (14):

$$\nu (h)^{-\gamma_h} = (c)^{-\gamma} q - \beta q c^{-\gamma} - s\frac{q}{R}\mu,$$

which, combined with (A.1), returns:

$$\begin{aligned} \nu (h)^{-\gamma_h} &= (c)^{-\gamma} q - \beta q (c)^{-\gamma} - s\frac{q}{R}(c)^{-\gamma}(1-\beta R), \\ &= (c)^{-\gamma} q \left[1 - \beta - s\frac{1}{R}(1-\beta R)\right]. \end{aligned} \tag{A.4}$$

Equations (A.3) and (A.4) provide the unique solutions for c and q. Conditional on these, closed-form solutions for μ and d follow from (A.1) and (A.2), respectively.

B Solution algorithm

We solve the model numerically, through Euler-equation iteration of the policy functions. The problem is non-standard in that a state-dependent inequality is introduced through the collateral constraint. As argued by Rendahl (2015), solution searches can, in such cases, be divergent, cyclical, or even non-convergent. We therefore follow Judd (1988), and introduce 'dampening' parameters in the updating of the policy functions. This implies that, at any update of a policy function, only a fraction of the new function will replace the old one. This fosters convergence. Our approach is based on Jeanne and Korinek (2019), adapted to an environment with a borrowing constraint involving the expected-future price of the collateral asset.

We first discretize the state variables d_{t-1}, e_t, and s_t such that $d_{t-1} \in \mathbf{d}_{t-1} \equiv [d_{\min}, \ldots, d_{\max}]^\mathsf{T}$, $e_t \in \mathbf{e}_t \equiv [e_{\min}, \ldots, e_{\max}]^\mathsf{T}$, $s_t \in \mathbf{s}_t \equiv [s_{\min}, \ldots, s_{\max}]^\mathsf{T}$. In the construction of the state vectors, we make sure that the model does not imply starvation for high initial debt combined with sufficiently adverse shocks. The discretization of the shocks relies on Rouwenhorst's (1995) method of approximating AR(1) processes by Markov chains with transition matrices, $\mathbf{P_e}$ and $\mathbf{P_s}$. We thereby follow Kopecky and Suen (2010), who find that this method best approximates very persistent processes, compared with other methods. To simplify notation and computation, we create a column vector of all shock combinations, $\mathbf{z}_t \equiv \text{vec}\big(\mathbf{s}_t\mathbf{e}_t^\mathsf{T}\big)$. The associated transition matrix for $\mathbf{z}_t$ is then given by $\Pi \equiv \mathbf{P_e} \otimes \mathbf{P_s}$, where $\otimes$ is the Kronecker product. We use 2,501 debt states and five states for each shock.

In the solution procedure, we construct a matrix of all state combinations, $\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}$, and seek solutions for policy functions yielding matrices $c\left(\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}\right)$, $q\left(\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}\right)$, $d\left(\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}\right)$, and $\mu\left(\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}\right)$, which satisfy the equilibrium conditions. Note that, in any state, we have either $\mu_t = 0$ or $\mu_t > 0$. We refer to these two cases as the *unconstrained* and the *constrained* regime, respectively. In each iteration, we solve the model in two blocks—one for each regime. This exploits the different structure of the solution in either regime. Subsequently, the policy matrices are appropriately merged before proceeding with the next iteration. The algorithm involves the following steps:

1. Make initial guesses $c^0\left(\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}\right)$ and $q^0\left(\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}\right)$.

2. Use (12) to obtain $d\left(\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}\right) = c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}\right) + R\widetilde{\mathbf{d}}_{t-1} - yf\left(\widetilde{\mathbf{e}}_t\right)$, where $\widetilde{\mathbf{d}}_{t-1}$ is a matrix of repeated columns of $\mathbf{d}_{t-1}$ conformable with $\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}$. The income shocks are used to construct the matrix $f\left(\widetilde{\mathbf{e}}_t\right)$, which has identical row vectors of the possible income-shock values, conformable with $\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}$.

3. Use $\mathbf{d}_t = d\left(\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}\right)$ to compute $\mathbf{c}_{t+1} = \widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\mathsf{T}\right)$ and $\mathbf{q}_{t+1} = \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\mathsf{T}\right)$ through column-wise interpolation on $\mathbf{d}_{t-1}$ and $c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}\right)$ and $q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}\right)$, respectively.

4. Derive the policy functions in the *unconstrained* regime:

   (a) By definition, $\mu^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}\right) = 0$.

   (b) From (13):

$$c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\mathsf{T}\right) = \left\{\beta R\left[\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\mathsf{T}\right)^{-\gamma}\Pi^\mathsf{T}\right]\right\}^{-1/\gamma}.$$

(c) From (14):

$$\begin{aligned} q^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) &= c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{\gamma} \\ &\circ\left\{\nu\left(h\right)^{-\gamma_h}+\beta\left[\left(\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\circ\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma}\right)\Pi^{\mathsf{T}}\right]\right\}, \end{aligned}$$

where $\circ$ denotes element-by-element multiplication.

(d) By (12), find $d^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)=c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)+R\widetilde{\mathbf{d}}_{t-1}-yf\left(\widetilde{\mathbf{e}}_t\right)$.

5. Derive the policy functions in the *constrained* regime:

(a) Let the matrix $\widetilde{\mathbf{s}}_t$ contain identical row vectors of the possible LTV-shock values, conformable with $\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}$. In each column of $\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}$ identify the states where:

$$d^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)>\left[\left(s+\widetilde{\mathbf{s}}_t\right)/R\right]\circ\left[\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}\right]h,$$

as these violate (4) and therefore characterize the constrained regime. For any matrix $\mathbf{X}_t$, denote by $\left[\mathbf{X}_t\right]^j$ the j^{th} column of $\mathbf{X}_t$ only consisting of such identified states.

(b) From (4), when the constraint binds:

$$\left[d^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j=\left(\left[s+\widetilde{\mathbf{s}}_t\right]^j/R\right)\circ\left[\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}\right]^j h,\quad \text{all } j.$$

(c) From (15):

$$\left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j=y\left[f\left(\widetilde{\mathbf{e}}_t\right)\right]^j-R\left[\widetilde{\mathbf{d}}_{t-1}\right]^j+\left[d^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j,\quad \text{all } j.$$

(d) From (13):

$$\left[\mu^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j=\left(\left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j\right)^{-\gamma}-\beta R\left[\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma}\Pi^{\mathsf{T}}\right]^j,\quad \text{all } j.$$

(e) From (14):

$$\begin{aligned} \left[q^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j &= \left(\left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j\right)^{\gamma} \\ &\circ\left\{\nu\left(h\right)^{-\gamma_h}+\beta\left[\left(\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\circ\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma}\right)\Pi^{\mathsf{T}}\right]^j\right. \\ &\left.+\left(\left[s+\widetilde{\mathbf{s}}_t\right]^j/R\right)\circ\left[\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma}\Pi^{\mathsf{T}}\right]^j\circ\left[\mu^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j\right\},\quad \text{all } j. \end{aligned}$$

6. An updated set of policy functions $c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$, $q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$, and the associated $d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$ and $\mu\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$, are built from the respective matrices found in the unconstrained and constrained regimes. Specifically, in the policy matrices derived for the unconstrained regime,

replace the values with the ones found in the constrained regime for the states identified in Step 5a.

7. If

$$\left\|\text{vec}\left[c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]-\text{vec}\left[c^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]\right\|_{\infty}<\varepsilon,$$

and

$$\left\|\text{vec}\left[q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]-\text{vec}\left[q^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]\right\|_{\infty}<\varepsilon,$$

where ε is some tolerance criterion, then stop (we use $\varepsilon = 10^{-8}$). Otherwise, update according to $c^{i+2}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)=\omega_c c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)+\left(1-\omega_c\right)c^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$ and $q^{i+2}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)=\omega_q q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)+\left(1-\omega_q\right)q^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$, where $0<\omega_c,\ \omega_q<1$ are dampening parameters, and go to 2.

Subsequently, the value function is computed. Start with a guess $V^0\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$. Then proceed as follows:

1. Use $d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$ to obtain $V_{t+1}=\widehat{V}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)$ through column-wise interpolation on $\mathbf{d}_{t-1}$ and $V^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$.

2. Compute:

$$V^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)=\frac{1}{1-\gamma}\left[c\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^{1-\gamma}+\frac{\nu}{1-\gamma_h}\left(h\right)^{1-\gamma_h}+\beta\widehat{V}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}.$$

3. If

$$\left\|\text{vec}\left[V^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]-\text{vec}\left[V^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]\right\|_{\infty}<\varepsilon,$$

where ε is the tolerance criterion, then stop. Otherwise, set $V^{i+2}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)=\omega_V V^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)+\left(1-\omega_V\right)V^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$, where $0<\omega_V<1$ is a dampening parameter, and go to 1.

C The welfare cost of business cycles

We aim at finding the value of λ that secures $\mathbb{E}\left[V\left(d_{t-1}, z_t\right)\right] = \mathbb{E}\left[\overline{V}\left(d_{t-1}\right)\right]$; i.e., indifference between the stochastic and non-stochastic economies. Using the definitions of the value functions, and defining λ as the percentage increase in the consumption path in the stochastic economy that secures indifference with respect to steady-state consumption:

$$
\begin{aligned}
& \mathbb{E}\left[\sum_{t=1}^{\infty}\beta^{t-1}\left(\frac{1}{1-\gamma}\left[\left(1+\lambda/100\right)c\left(d_{t-1}, z_t\right)\right]^{1-\gamma}+\frac{\nu}{1-\gamma_h}h^{1-\gamma_h}\right)\right] \qquad \text{(C.1)} \\
= \; & \mathbb{E}\left[\sum_{t=1}^{\infty}\beta^{t-1}\left(\frac{1}{1-\gamma}\left[\overline{c}\left(d_{t-1}\right)\right]^{1-\gamma}+\frac{\nu}{1-\gamma_h}h^{1-\gamma_h}\right)\right],
\end{aligned}
$$

where $\overline{c}\left(d_{t-1}\right)$ is the policy function for consumption under certainty. From (C.1) we readily obtain:

$$
\left(1+\lambda/100\right)^{1-\gamma}\mathbb{E}\left[\sum_{t=1}^{\infty}\beta^{t-1}\frac{1}{1-\gamma}\left[c\left(d_{t-1}, z_t\right)\right]^{1-\gamma}\right] = \mathbb{E}\left[\sum_{t=1}^{\infty}\beta^{t-1}\frac{1}{1-\gamma}\left[\overline{c}\left(d_{t-1}\right)\right]^{1-\gamma}\right],
$$

and, therefore:

$$
\begin{aligned}
\left(1+\lambda/100\right)^{1-\gamma} &= \frac{\mathbb{E}\left[\sum_{t=1}^{\infty}\beta^{t-1}\frac{1}{1-\gamma}\left[\overline{c}\left(d_{t-1}\right)\right]^{1-\gamma}\right]}{\mathbb{E}\left[\sum_{t=1}^{\infty}\beta^{t-1}\frac{1}{1-\gamma}\left[c\left(d_{t-1}, z_t\right)\right]^{1-\gamma}\right]}, \qquad \text{(C.2)} \\
&= \frac{\mathbb{E}\left[\overline{V}\left(d_{t-1}\right)\right]-u^h}{\mathbb{E}\left[V\left(d_{t-1}, z_t\right)\right]-u^h},
\end{aligned}
$$

where the second line in (C.2) follows from the definitions of the value functions and u^h. From (C.2), we immediately recover the unconditional welfare measure, as desired.

Conditional welfare As for the conditional welfare measure, $\lambda^c\left(d_{t-1}, z_t\right)$, this satisfies:

$$
\begin{aligned}
& \mathbb{E}_t\left[\sum_{s=t}^{\infty}\beta^{s-t}\left(\frac{1}{1-\gamma}\left[\left(1+\lambda^c\left(d_{t-1}, z_t\right)/100\right)c\left(d_{s-1}, z_s\right)\right]^{1-\gamma}+\frac{\nu}{1-\gamma_h}h^{1-\gamma_h}\right)\right] \\
= \; & \mathbb{E}_t\left[\sum_{s=t}^{\infty}\beta^{s-t}\left(\frac{1}{1-\gamma}\left[\overline{c}\left(d_{s-1}\right)\right]^{1-\gamma}+\frac{\nu}{1-\gamma_h}h^{1-\gamma_h}\right)\right]. \qquad \text{(C.3)}
\end{aligned}
$$

Similar manipulations to those involving (C.1) readily yield (17).

Ergodicity The unconditional welfare measure depends on the ergodic distribution of debt in the stochastic economy, as well as on the steady state in the deterministic economy. To perform a robustness analysis, a comparison of the economies under the same ergodic distribution of debt is warranted. Specifically, the deterministic economy value function in (16) can be evaluated using the ergodic distribution of the stochastic economy. To perform a welfare comparison based on the same benchmark—i.e., the same ergodic distribution of debt—we devise:

$$\lambda^E = 100 \times \left\{ \left(\frac{\mathbb{E}^S[\bar{V}\,(d_{t-1})] - u^h}{\mathbb{E}^S[V\,(d_{t-1}, z_t)] - u^h} \right)^{\frac{1}{1-\gamma}} - 1 \right\}, \tag{C.4}$$

where, $\mathbb{E}^S\,(X) = \int_X X \cdot \mathrm{d}F^s\,(X)$ and $\mathrm{d}F^s$ is the ergodic density of the stochastic economy. Such comparison returns a welfare loss of -0.2063%.

D Additional figures

Sample of simulated data Figure D.1 provides an illustration of the dynamics of some selected variables of our model, for different sizes of the shocks to the economy. We consider a sequence of 20 periods. The realizations of the two shock processes, e_t and s_t, are shown in the bottom row, while the remaining panels report the dynamics of some key endogenous variables. We consider two cases: The black lines represent our baseline economy, while the green lines correspond to a case where the standard deviations of both shocks have been multiplied by a factor of 3 (i.e., $k = 3$, using the notation of Section 6).

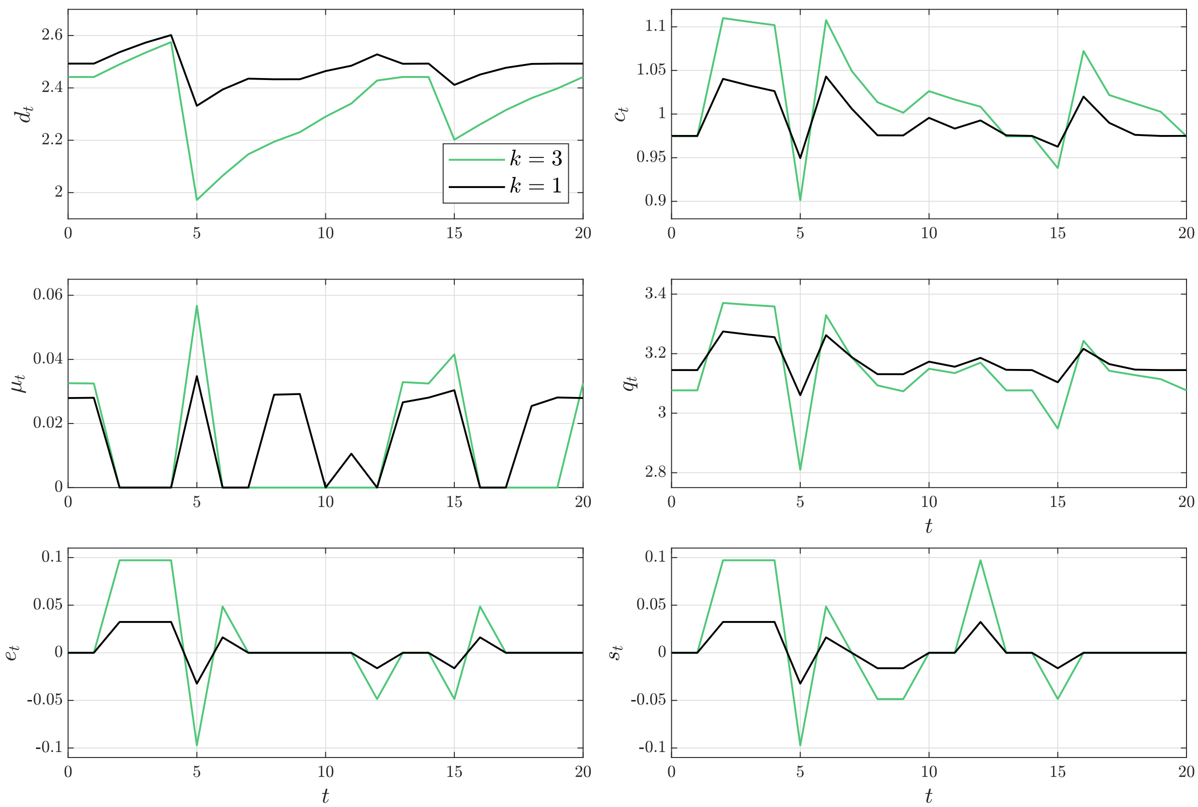

Figure D.1: Sample of simulated paths of selected variables for different shock sizes. In each panel, the black line reports the path of a given variable conditional on the baseline calibration of the model, while the green line reports the path of the same variable conditional on larger shocks ($k = 3$). The variables are debt (top left panel), consumption (top right), shadow value of borrowing constraint (μ_t, middle left), asset price (middle right), exogenous component of income process (bottom left), exogenous component of borrowing constraint (bottom right).

As can be appreciated from the dynamics of μ_t (second row, left panel), households become financially unconstrained more often when shocks are larger. It is noteworthy, for example, that the negative income shock occurring in period 8 (and persisting in period 9) makes the households financially constrained in the case of $k = 1$, but not in the case of $k = 3$, despite the fact that the shock is three times larger in the latter case. The reason is that, in anticipation of future large shocks, households act in a precautionary manner by contracting less debt in the preceding periods (see top row, left panel). This allows them to enjoy higher consumption for a number of periods around this shock (top row, right panel). More generally, the observed paths showcase the insights obtained from the right panel of Figure 2: Larger shocks are associated with lower debt levels and a higher level of consumption on average, although large negative shock realizations occasionally force households into quite dramatic reductions of consumption (as in period 5, for example).

More on the role of different shocks The right panel of Figure 2 in the main text reported the pattern of average consumption and debt as the size of the shocks hitting the economy increases. We now report the results of a similar exercise, but where only the standard deviation of one shock at a time is raised, with the standard deviation of the other shock set to zero (i.e., the same experiment as in Figure 3). The results of this exercise are shown in Figure D.2. The figure makes it clear that the patterns reported in Figure 2 are confirmed, conditional on either of the two shocks in isolation. In both cases, we observe a gradual decline in average debt alongside a gradual increase in average consumption.

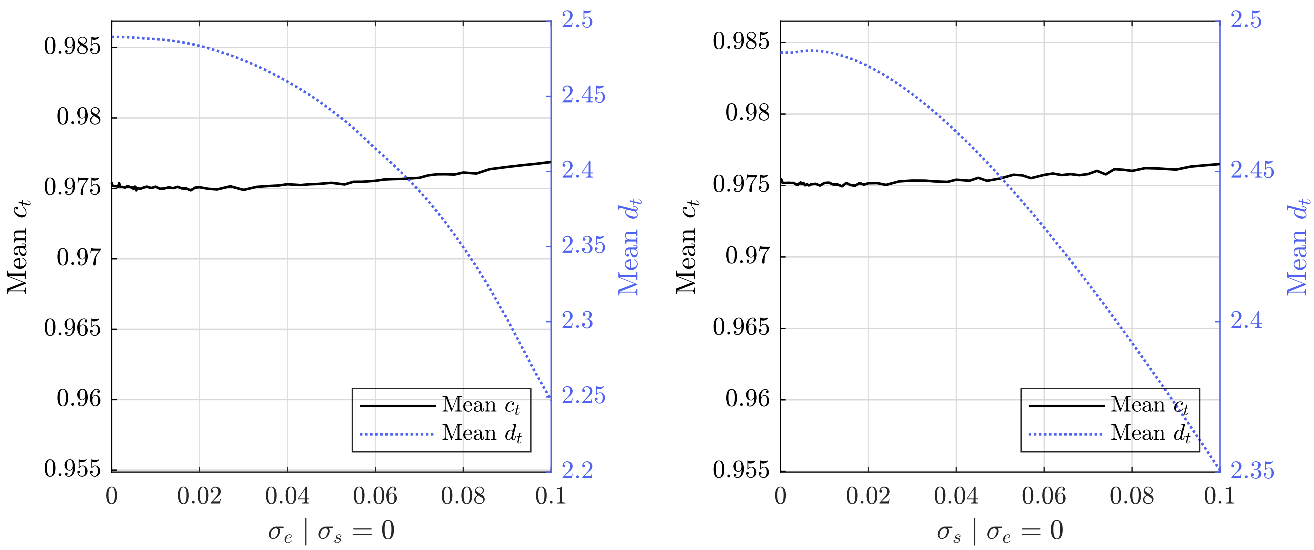

Figure D.2: Varying uncertainty in the baseline economy. In each panel, the continuous (black) line reports the pattern of average consumption, and the dotted (blue) line that of average debt for different standard deviations of a given shock, conditional on the other shock being switched off. All the other parameters are at their baseline values.

E Equivalence between the decentralized and the constrained-efficient equilibrium

We now consider the constrained-efficient equilibrium (CEE) attained by a social planner (SP). Thus, we demonstrate that the resulting policy functions are equivalent to those obtained in the decentralized equilibrium (DE), in the vein of Ottonello *et al.* (2022). Finally, we confirm our theoretical result in a numerical exercise.

E.1 Constrained-efficient equilibrium

To isolate the role played by the pecuniary externality associated with the households' optimal choice of debt, we first consider an optimization problem where households choose consumption of durable and nondurable goods, while the SP chooses the optimal amount of debt for a given period, taking her future periods' choices as given.

Household problem

The household problem can be written as:

$$\max_{\{c_t,h_t\}} \mathbb{E}_1 \left[\sum_{t=1}^{\infty} \beta^{t-1} \left(\frac{1}{1-\gamma} c_t^{1-\gamma} + \frac{\nu}{1-\gamma_h} h_t^{1-\gamma_h}\right)\right],$$

subject to:

$$c_t + q_t \left(h_t - h_{t-1}\right) = yf\left(e_t\right) + T_t,$$
$$d_t \leq \left(s_t + s\right) \frac{\mathbb{E}_t\left[q_{t+1}\right] h_t}{R},$$

where $T_t = d_t - Rd_{t-1}$.

The Lagrangian reads as:

$$\mathcal{L} = \mathbb{E}_1 \left[\sum_{t=1}^{\infty} \beta^{t-1} \left(\frac{1}{1-\gamma} c_t^{1-\gamma} + \frac{\nu}{1-\gamma_h} h_t^{1-\gamma_h} + \Lambda_t \left[yf\left(e_t\right) + T_t - c_t - q_t\left(h_t - h_{t-1}\right)\right] + \mu_t \left[\left(s_t + s\right)\frac{\mathbb{E}_t\left[q_{t+1}\right] h_t}{R} - d_t\right]\right)\right].$$

The corresponding FOCs are:

$$c_t^{-\gamma} - \Lambda_t = 0,$$
$$\nu h_t^{-\gamma_h} - q_t\Lambda_t + \mu_t \left(s_t + s\right) \frac{\mathbb{E}_t\left[q_{t+1}\right]}{R} + \beta\mathbb{E}_t\left(\Lambda_{t+1} q_{t+1}\right) = 0,$$

which combine into:

$$c_t^{-\gamma} q_t = \nu h_t^{-\gamma_h} + \beta \mathbb{E}_t \left(c_{t+1}^{-\gamma} q_{t+1} \right) + (s + s_t) \frac{\mathbb{E}_t [q_{t+1}]}{R} \mu_t. \tag{E.1}$$

This equation is equivalent to the Euler equation of durables for households in the decentralized setting; see (10). This will enter into the SP's problem, to which we now turn.

Constrained-efficient allocation

The SP faces the following problem:

$$V(d_{t-1}, z_t) = \max_{\{c_t, d_t\}} \left\{ \frac{1}{1-\gamma} c_t^{1-\gamma} + \frac{\nu}{1-\gamma_h} h^{1-\gamma_h} + \beta \mathbb{E}_t [V(d_t, z_{t+1})] \right\} \text{ s.t.} \tag{E.2}$$

$$c_t = d_t + y f(e_t) - R d_{t-1}, \tag{E.3}$$

$$d_t \leq (s + s_t) \frac{\mathbb{E}_t [q_{t+1}] h}{R}, \tag{E.4}$$

$$q_t = \frac{\nu h^{-\gamma_h} + \mu_t (s + s_t) \frac{\mathbb{E}[q_{t+1}]}{R} + \beta \mathbb{E}_t \left[c_{t+1}^{-\gamma} q_{t+1} \right]}{c_t^{-\gamma}}, \tag{E.5}$$

$$\mu_t = \begin{cases} 0 & \text{if } d_t < (s + s_t) \frac{\mathbb{E}_t[q_{t+1}] h}{R}, \\ c_t^{-\gamma} - R\beta \mathbb{E}_t \left(c_{t+1}^{-\gamma} \right) & \text{otherwise,} \end{cases} \tag{E.6}$$

where the market-clearing condition $h_t = h$ has been imposed. Setting up the Lagrangian:

$$\begin{aligned} \mathcal{L} = & \frac{1}{1-\gamma} c_t^{1-\gamma} + \frac{\nu}{1-\gamma_h}^{1-\gamma_h} + \beta \mathbb{E}_t [V(d_t, z_{t+1})] + \Lambda_t [d_t + y f(e_t) - R d_{t-1} - c_t] \\ & + \mu_t^* \left[(s + s_t) \frac{\mathbb{E}_t [q_{t+1}] h}{R} - d_t \right]. \end{aligned}$$

The FOC's w.r.t. c_t and d_t and the Envelope Condition are, respectively:

$$c_t^{-\gamma} - \Lambda_t = 0,$$

$$\beta \mathbb{E}_t \left[\frac{\partial V(d_t, z_{t+1})}{\partial d_t} \right] + \Lambda_t - \mu_t^* \left[1 - (s + s_t) \frac{\mathbb{E}_t \left[\frac{\partial q_{t+1}}{\partial d_t} \right] h}{R} \right] = 0,$$

$$\frac{\partial V(d_{t-1}, z_t)}{\partial d_{t-1}} = -\Lambda_t R.$$

Combining these yields:

$$c_t^{-\gamma} = R\beta \mathbb{E}_t \left[c_{t+1}^{-\gamma} \right] + \mu_t^* \left[1 - (s + s_t) \frac{\mathbb{E}_t \left[\frac{\partial q_{t+1}}{\partial d_t} \right] h}{R} \right]. \tag{E.7}$$

E.2 Proof of Proposition 1

Ottonello *et al.* (2022) demonstrate that the DE is constrained-efficient by appealing to the fact that the Euler equation for models with collateral constraints based on future prices, ($\mathbb{E}q_{t+1}$, in our case), is equivalent to that obtained by the SP in the CEE. To grasp such equivalence in our setting, consider the Euler equation in the DE presented in Section 2:

$$c_t^{-\gamma} = R\beta\mathbb{E}_t\left[c_{t+1}^{-\gamma}\right] + \mu_t,$$

and compare it to the expression obtained for the CEE case (i.e., E.7):

$$c_t^{-\gamma} = R\beta\mathbb{E}_t\left[c_{t+1}^{-\gamma}\right] + \mu_t^*\left[1 - (s + s_t)\frac{\mathbb{E}_t\left[\frac{\partial q_{t+1}}{\partial d_t}\right]h}{R}\right].$$

Given some regularity conditions, these equations are "equivalent up to a scalar"; i.e., the resulting policy functions are the same, except those accounting for the shadow prices (μ_t and μ_t^*). This leads us to a confirmation of the result of Ottonello *et al.* (2022) in Proposition 1, which implies that the DE and the corresponding policy functions coincide with the CEE and its policy functions, respectively.

To prove this, we start by formulating the following lemma:

Lemma 1. *Let $\mathcal{Z}$ be the discretized state space. For the DE and with the baseline calibration, the conditional expectation, $\mathbb{E}_t\left[\frac{\partial q_{t+1}}{\partial d_t}\right]$, is negative for $\forall d_t, z_{t+1} \in \mathcal{Z}$.*

Proof of Lemma 1. We resort to a numerical proof to demonstrate that q_t is decreasing in d_{t-1} for all combinations of z_t; cfr. Figure E.1. As a result of this property, $\frac{\partial q_{t+1}}{\partial d_t} < 0$. In turn, this implies that:

$$\mathbb{E}_t\left[\frac{\partial q_{t+1}}{\partial d_t}\right] = \sum_{i'}\left[\frac{\partial q_{t+1}}{\partial d_t}\right]_{i,i'}\Pi_{i,i'} < 0 \qquad \forall z_t, d_{t-1} \in \mathcal{Z},$$

where $\Pi_{i,i'}$ is the i, i' entry of the Markov matrix, Π, of the exogenous shocks with respect to states i and i', and where $\Pi_{i,i'} \geq 0$ and $\sum_{i'}\Pi_{i,i'} = 1$. □

We are now ready to prove Proposition 1.

Proof of Proposition 1. Assume that the consumption and debt policy functions, as well as the identities involving q_t and μ_t in the CEE coincide with their homologous in the DE. When comparing the Euler equation in the CEE with that in the DE, we see that these are equivalent up to a scalar. As a result, we can construct a mapping from the DE multiplier, μ_t, to its homologous in the CEE, μ_t^*:

$$\mu_t^* = \left[1 - (s + s_t)\frac{\mathbb{E}_t\left[\frac{\partial q_{t+1}}{\partial d_t}\right]h}{R}\right]^{-1}\mu_t.$$

Since the scalar function, $\left[1 - (s + s_t)\mathbb{E}_t\left[\frac{\partial q_{t+1}}{\partial d_t}\right]h/R\right]^{-1}$, is non-negative by virtue of Lemma 1, then $\mu_t^* \geq 0$ satisfies both a complementary slackness condition and the Euler equation in the

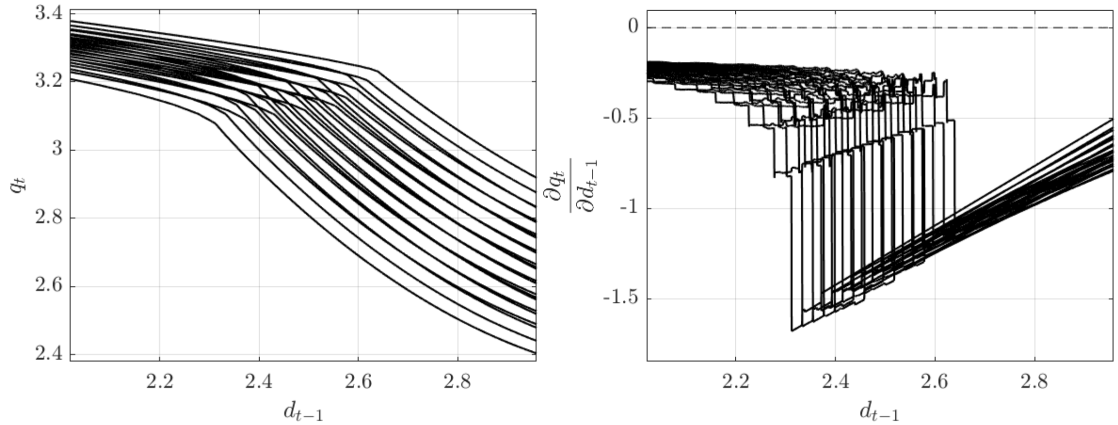

Figure E.1: Numerical policy functions of q_t (left) and $\partial q_t/\partial d_{t-1}$ (right).

SP's problem. By using the mapping, it is possible to traverse from the DE equations to their CEE counterparts, thus proving that they coincide. $\square$

E.3 Numerical implementation

The DE-CEE equivalence can be numerically verified by solving the SP's problem and comparing the CEE to the DE. To this end, we describe the solution method, which is an extension of the solution method employed for the DE, as described in Appendix B. The major difference is that the derivatives of the policy functions enter the equations of interest.

Once again, we construct matrices of all state combinations, $\mathbf{d}_{t-1}\mathbf{z}_t^\intercal$, and seek the following policy functions in matrix form: $c\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right), q\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right), d\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right), \mu^*\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right)$, and $\mu\left(\mathbf{d}_{t-1}\mathbf{z}_t^\intercal\right)$.

These five policy functions must satisfy the following five equations:

$$
\begin{aligned}
c\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)^{-\gamma} &= \beta R\left[\left(c\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\top\right)\right)^{-\gamma}\Pi^\top\right] + \mu^*\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right) \\
&\qquad \circ\left[1-\frac{(s+\widetilde{\boldsymbol{s}})\,h}{R}\circ\left[\frac{\partial q\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\top\right)}{\partial d\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)}\Pi^\top\right]\right], \qquad \text{(E.8)}\\
d\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right) &= c\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right) - yf\left(\widetilde{\boldsymbol{e}}\right) + R\widetilde{\mathbf{d}}_{t-1}, \qquad \text{(E.9)}\\
0 &= \mu^*\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)\left[\frac{(s+\widetilde{\boldsymbol{s}})\,h}{R}\circ q\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\top\right)\Pi^\top - d\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)\right], \qquad \text{(E.10)}\\
q\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right) &= c\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)^{\gamma}\circ \\
&\qquad \left[\nu h^{-\gamma_h} + \mu\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)\circ\frac{(s+\widetilde{\boldsymbol{s}})}{R}\circ q\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\top\right)\Pi^\top\right.\\
&\qquad \left.+\beta q\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\top\right)\circ c\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\top\right)^{-\gamma}\Pi^\top\right], \qquad \text{(E.11)}\\
\mu\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right) &= \begin{cases} 0 \\ \text{if } \frac{(s+\widetilde{\boldsymbol{s}})h}{R}\circ q\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\top\right)\Pi^\top > d\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right), \\ c\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)^{-\gamma} - \beta R\left[\left(c\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\top\right)\right)^{-\gamma}\Pi^\top\right], \\ \text{if } \frac{(s+\widetilde{\boldsymbol{s}})h}{R}\circ q\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\top\right)\Pi^\top \le d\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right). \end{cases} \qquad \text{(E.12)}
\end{aligned}
$$

Note, $\widetilde{\boldsymbol{s}}$, $\widetilde{\mathrm{d}}_{t-1}$, and $\widetilde{\boldsymbol{e}}$ are defined as in Appendix B. The solution method proceeds in the following steps:

1. Generate a discrete grid of the state space and use the decentralized policy functions as an initial guess for the policy functions: $c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right), q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right), d^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right), \mu^{i*}\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)$, and $\mu^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)$.
2. Use (E.9) to obtain $d\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)$:

$$
d\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right) = c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right) + R\widetilde{\mathbf{d}}_{t-1} - yf\left(\widetilde{\mathbf{e}}_t\right).
$$

3. Compute future values:
   (a) Apply $\mathbf{d}_t = d\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)$ to interpolate on $c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)$ and $q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)$ to obtain $\mathbf{c}_{t+1} = \widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\top\right)$ and $\mathbf{q}_{t+1} = \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^\top\right)$.
   (b) Obtain a derivative of $q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)$ with respect to $\mathbf{d}_{t-1}$ by central finite difference:

$$
\left[\partial q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)/\partial d_{t-1}\right]_{i_d,i_z} \approx \frac{\left[q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)\right]_{i_d+1,i_z} - \left[q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)\right]_{i_d-1,i_z}}{\left[\widetilde{\mathbf{d}}_{t-1}\right]_{i_d+1,i_z} - \left[\widetilde{\mathbf{d}}_{t-1}\right]_{i_d-1,i_z}}.
$$

   Here, i_d denotes the index in the debt dimension, and i_z is the index in the exogenous shock dimension of the matrices. Then interpolate the derivative using $\mathbf{d}_t$ and the result is denoted as $\frac{\partial q\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)}{\partial d\left(\mathbf{d}_{t-1}\mathbf{z}_t^\top\right)}$.

4. Derive the unconstrained regime:

   (a) As a result, $\mu^{*uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \mu^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \mathbf{0}$.

   (b) From (E.8) consumption is pinned down:

   $$c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \left\{\beta R\left[\left(\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\right)^{-\gamma}\Pi^{\mathsf{T}}\right]\right\}^{\frac{-1}{\gamma}}.$$

   (c) From (E.11) we obtain:

   $$q^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{\gamma}\circ\left[\nu h^{-\gamma_h} + \beta\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\circ\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma}\Pi^{\mathsf{T}}\right].$$

   (d) From (E.9) we obtain:

   $$d^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) - yf\left(\widetilde{\boldsymbol{e}}\right) + R\widetilde{\mathbf{d}}_{t-1}.$$

5. Derive the policy functions in the constrained regime:

   (a) Identify the constrained regime. The following inequality identifies the states in $\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}$ where the constraint is binding:

   $$d^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) > \left(s+\widetilde{\boldsymbol{s}}\right)/Rh\circ\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}.$$

   Based on the inequality, an identifier is constructed such that for any matrix, $\mathbf{X}_t$, denote by $[\mathbf{X}_t]^j$ the j^{th} column of $\mathbf{X}_t$ only consisting of such identified states.

   (b) From (E.4) the constraint binds, and debt is obtained:

   $$\left[d^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \left[\frac{\left(s+\widetilde{\boldsymbol{s}}\right)h}{R}\circ\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}\right]^j, \quad \forall j.$$

   (c) From (E.9) consumption is:

   $$\left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \left[d^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) + yf\left(\widetilde{\boldsymbol{e}}\right) - R\widetilde{\mathbf{d}}_{t-1}\right]^j, \quad \forall j.$$

   (d) The Lagrange multiplier from the decentralized equilibrium is obtained with (E.12):

   $$\left[\mu^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma} - \beta R\left[\left(\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\right)^{-\gamma}\Pi^{\mathsf{T}}\right]\right]^j, \quad \forall j.$$

(e) The asset price is pinned down by (E.11):

$$\left[q^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{\gamma}\right]^j \circ$$
$$\left[\nu h^{-\gamma_h} + \mu^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) \circ \frac{(s+\widetilde{\boldsymbol{s}})}{R} \circ \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}\right.$$
$$\left.+\beta\left[\widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right) \circ \widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma}\Pi^{\mathsf{T}}\right]\right]^j, \quad \forall j.$$

(f) The Lagrange multiplier of the SP's problem is found from (E.8):

$$\left[\mu^{con*}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \left[\left[1 - \frac{(s+\widetilde{\boldsymbol{s}})\,h}{R} \circ \left[\frac{\partial q\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)}{\partial d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)}\Pi^{\mathsf{T}}\right]\right]^{-1}\right]^j$$
$$\circ\left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma} - \beta R\left[\left(\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\right)^{-\gamma}\Pi^{\mathsf{T}}\right]\right]^j, \quad \forall j.$$

6. A new set of policy functions are now constructed from the constrained and the unconstrained regime, using the identifiers:

$$c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), d^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), \mu^{*i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), \mu^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right).$$

7. Convergence depends on the following metrics:

$$\left\|\text{vec}\left[c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right] - \text{vec}\left[c^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]\right\|_{\infty} < \varepsilon,$$
$$\left\|\text{vec}\left[q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right] - \text{vec}\left[q^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]\right\|_{\infty} < \varepsilon.$$

Here, ε is a tolerance criterion. If the conditions are satisfied, then stop. If not, update the policy functions according to:

$$c^{i+2}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \omega_c c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) + (1-\omega_c)\,c^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right),$$
$$q^{i+2}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \omega_q q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) + (1-\omega_q)\,q^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right),$$

where ω_c and ω_q are weights. Reset $i+2$ to i and return to Step 2.

E.4 Numerical evidence

We are now ready to look at the numerical proof. To this end, we first compare the policy functions obtained in the DE with those obtained in the CEE, each of which is reported in Figure E.2. To evaluate the differences between the two, we consider the following metric:

$$\left|c^{DE} - c^{CEE}\right|/c^{DE} = \left|c^{DE}\left(d_{t-1}, z_t\right) - c^{SP}\left(d_{t-1}, z_t\right)\right|/c^{DE}\left(d_{t-1}, z_t\right),$$
$$\left|q^{DE} - q^{CEE}\right|/q^{DE} = \left|q^{DE}\left(d_{t-1}, z_t\right) - q^{CEE}\left(d_{t-1}, z_t\right)\right|/q^{DE}\left(d_{t-1}, z_t\right).$$

From Figure E.3, we see that these are virtually identical up to the sixth decimal; i.e., they never deviate from each other by more than 0.0001 of a percent.

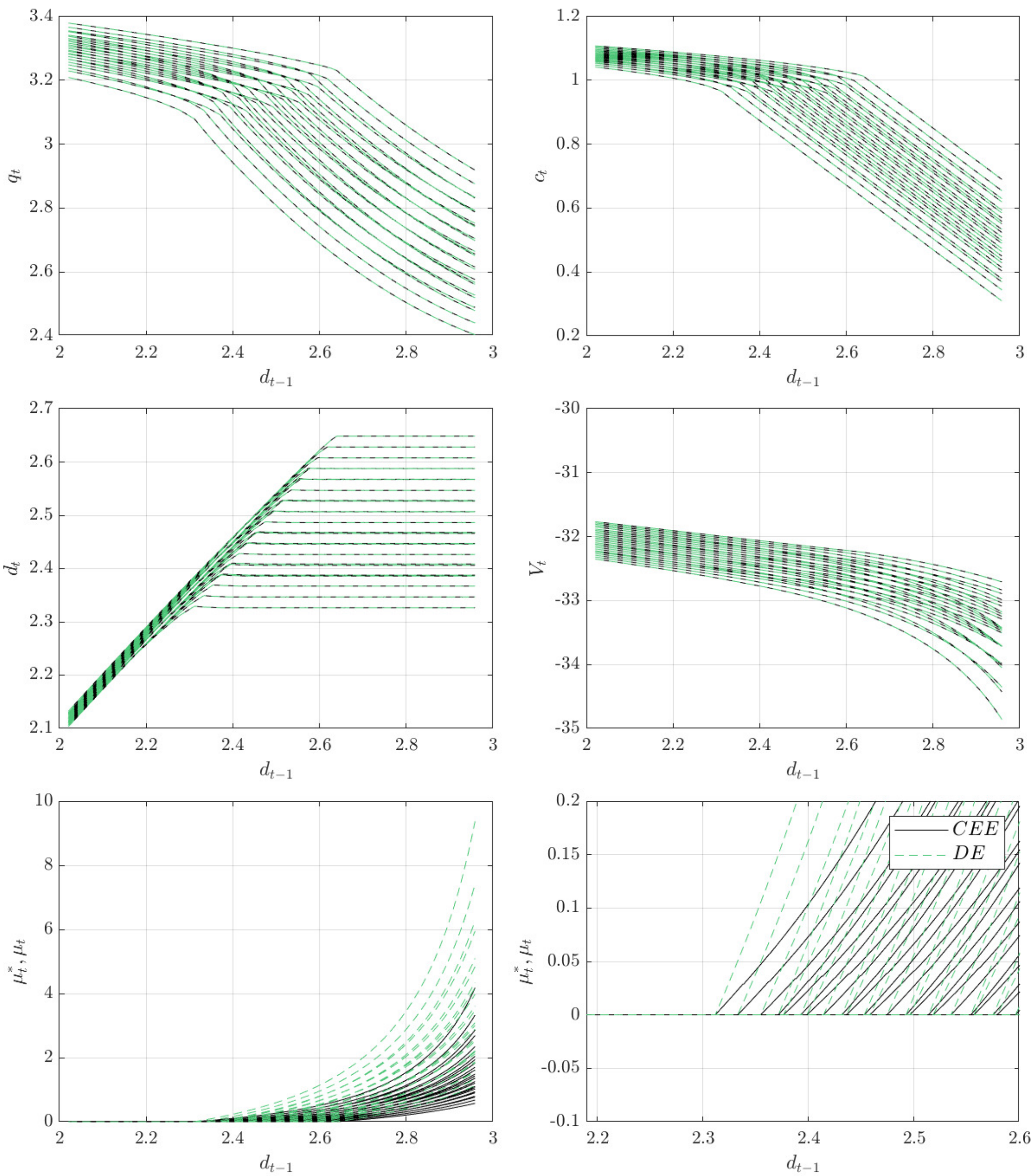

Figure E.2: Policy functions (solid line: CEE; dotted green line: DE).

The next step is to compute the loss of welfare from uncertainty in each of the two cases. We first compute the unconditional loss obtained by the SP in the CEE, which is $\lambda = -0.2128\%$, i.e., exactly the same as that obtained in the DE. We then turn to study how this gain changes with the size of the shocks. The results of this exercise are reported in Figure 4 in the main text. As is clear

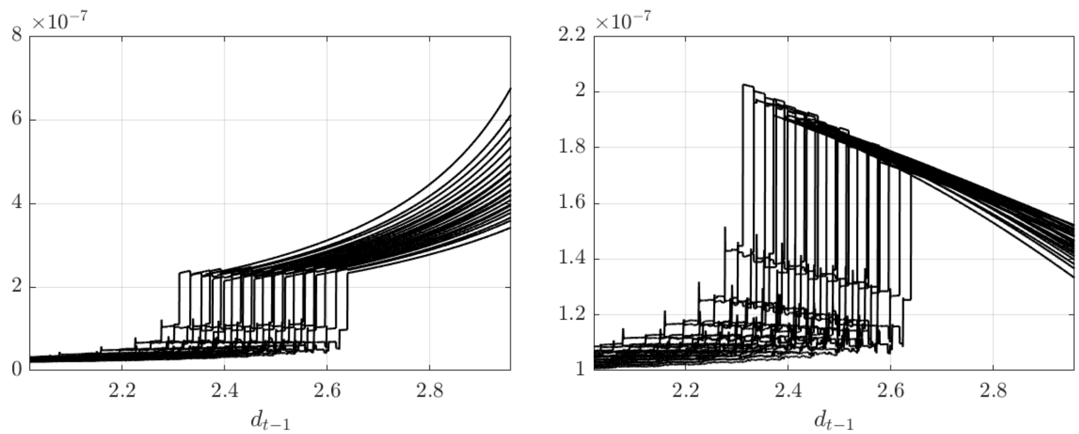

Figure E.3: Numerical equivalence. The figure plots the absolute percentage deviations: $\left|c^{DE} - c^{CEE}\right| / c^{DE}$ (left) and $\left|q^{DE} - q^{CEE}\right| / q^{DE}$ (right).

from the second row of that figure, there is no difference between the welfare attained in each of the two cases, thus confirming our analytical result.

F Current-period asset price in the collateral constraint

We now turn our attention to an economy in which the collateral constraint faced by households depends on the current-period asset price, instead of the expected-future asset price, as considered so far. Ottonello *et al.* (2022) have shown that, in this case, the equivalence result between the DE and the CEE breaks down. In this Appendix, we first confirm this result in an analytical context. We then resort to numerical exercises to conduct welfare comparisons between the two equilibria.

F.1 Decentralized equilibrium

We consider the same setup as outlined in Section 2, with the only difference that the collateral constraint (4) is now replaced by:

$$d_t \leq (s+s_t)\,\frac{q_t h_t}{R}, \qquad t = 1, 2, ..., \infty\,. \tag{F.1}$$

As before, the households maximize lifetime utility choosing c_t, h_t, and d_t. We can set up the corresponding Lagrangian:

$$\begin{aligned}\mathcal{L} = \mathbb{E}_1 \Bigg[\sum_{t=1}^{\infty} \beta^{t-1} \Bigg(\frac{c_t^{1-\gamma}}{1-\gamma} + \nu \frac{h_t^{1-\gamma_h}}{1-\gamma_h} \ + \Lambda_t \left[y f\left(e_t\right) - R d_{t-1} - c_t - q_t\left(h_t - h_{t-1}\right) + d_t\right] \\ + \ \mu_t \left[\left(s_t + s\right) \frac{q_t h_t}{R} - d_t \right] \Bigg)\Bigg]\end{aligned}$$

and derive the corresponding FOC's:

$$\begin{aligned} c_t^{-\gamma} - \Lambda_t = 0, \\ \beta^{t-1}\left(\Lambda_t - \mu_t\right) - \beta^t \mathbb{E}_t\left[\left(\Lambda_{t+1} R\right)\right] = 0, \\ \beta^{t-1} \nu h_t^{-\gamma_h} + \beta^{t-1} \Lambda_t\left[-q_t\right] + \beta^{t-1} \mu_t \left[\left(s_t + s\right) \frac{q_t}{R}\right] + \beta^t \mathbb{E}_t\left[\left(\Lambda_{t+1}\left(-\right) q_{t+1}\left(-1\right)\right)\right] = 0, \end{aligned}$$

which can be combined to yield:

$$\begin{aligned} c_t^{-\gamma} &= \mu_t + \beta R \mathbb{E}_t\left[c_{t+1}^{-\gamma}\right], \\ q_t &= \frac{\nu h_t^{-\gamma_h} + \beta \mathbb{E}_t\left[q_{t+1} c_{t+1}^{-\gamma}\right]}{c_t^{-\gamma} - \mu_t \frac{(s_t+s)}{R}}. \end{aligned}$$

As in the main text, we assume that durables are in fixed supply, $h = 1$. We can then write the equilibrium conditions compactly as:

$$c_t + Rd_{t-1} = yf(e_t) + d_t, \tag{F.2}$$

$$0 = \mu_t \left[(s_t + s) \frac{q_t h}{R} - d_t \right], \tag{F.3}$$

$$c_t^{-\gamma} = \mu_t + \beta R \mathbb{E}_t \left(c_{t+1}^{-\gamma} \right), \tag{F.4}$$

$$q_t = \frac{\nu h^{-\gamma_h} + \beta \mathbb{E}_t \left[c_{t+1}^{-\gamma} q_{t+1} \right]}{c_t^{-\gamma} - \frac{(s_t+s)}{R}\mu_t}. \tag{F.5}$$

F.2 Constrained-efficient equilibrium

We now derive the CEE. As in Appendix E.1, we consider a situation in which the households choose the consumption of durable and nondurable goods, while the SP chooses the optimal amount of debt for a given period, taking her future periods' choices as given.

Consider the households' problem, which is to choose $\{c_t, h_t\}$ while taking $\{q_t, T_t\}$ as given. The problem can then be written as:

$$\max_{\{c_t,h_t\}} \mathbb{E}_1 \left[\sum_{t=1}^{\infty} \beta^{t-1} \frac{1}{1-\gamma} c_t^{1-\gamma} + \nu \frac{1}{1-\gamma_h} h_t^{1-\gamma_h} \right]$$

subject to:

$$c_t + q_t (h_t - h_{t-1}) = yf(e_t) + T_t,$$
$$d_t \le (s + s_t) \frac{q_t h_t}{R}.$$

The resulting first-order conditions for c_t and h_t can be collapsed into:

$$q_t = \frac{\nu h_t^{-\gamma_h} + \beta \mathbb{E}_t \left[c_{t+1}^{-\gamma} q_{t+1} \right]}{c_t^{-\gamma} - \mu_t \frac{(s_t+s)}{R}}, \tag{F.6}$$

which will again serve as a constraint in the SP's optimization problem. This is known as the *implementability constraint*. Imposing the market clearing condition for durable goods, the SP's optimization problem reads as:

$$V\left(d_{t-1}, z_t\right) = \max_{\{c_t, d_t\}} \left\{ \frac{1}{1-\gamma} c_t^{1-\gamma} + \frac{\nu}{1-\gamma_h} h^{1-\gamma_h} + \beta \mathbb{E}_t \left[V\left(d_t, z_{t+1}\right)\right] \right\} \text{ s.t.} \tag{F.7}$$

$$c_t = d_t + y f\left(e_t\right) - R d_{t-1}, \tag{F.8}$$

$$d_t \leq \left(s + s_t\right) \frac{q_t}{R} h, \tag{F.9}$$

$$q_t = \frac{\nu h^{-\gamma_h} + \beta \mathbb{E}_t \left[c_{t+1}^{-\gamma} q_{t+1} \right]}{c_t^{-\gamma} - \mu_t \frac{(s+s_t)}{R}}, \tag{F.10}$$

$$\mu_t = \begin{cases} 0 & \text{if } d_t \leq \left(s + s_t\right) \frac{q_t}{R} h, \\ c_t^{-\gamma} - R\beta \mathbb{E}_t \left[c_{t+1}^{-\gamma} \right] & \text{otherwise.} \end{cases} \tag{F.11}$$

In this case, we choose to rewrite the problem exclusively in terms of d_t, after inserting for c_t from the budget constraint. In addition, we impose that $q\left(d_{t-1}, z_t\right)$ is Markovian. Thus, writing the Lagrangian yields:

$$\begin{aligned} \mathcal{L} = & \frac{1}{1-\gamma} \left(d_t + y f\left(e_t\right) - R d_{t-1} \right)^{1-\gamma} + \frac{\nu}{1-\gamma_h} h^{1-\gamma_h} + \beta \mathbb{E}_t \left[V\left(d_t, z_{t+1}\right) \right] \\ & + \mu_t^* \left[\left(s + s_t\right) \frac{q\left(d_{t-1}, z_t\right) h}{R} - d_t \right], \end{aligned}$$

and the FOC's with respect to d_t and the Envelope condition are:

$$0 = \left(d_t + y f\left(e_t\right) - R d_{t-1} \right)^{-\gamma} + \beta \mathbb{E}_t \left[\frac{\partial V\left(d_t, z_{t+1}\right)}{\partial d_t} \right] - \mu_t^*, \tag{F.12}$$

$$\frac{\partial V\left(d_{t-1}, z_t\right)}{\partial d_{t-1}} = -R \left(d_t + y f\left(e_t\right) - R d_{t-1} \right)^{-\gamma} + \mu_t^* \frac{\left(s + s_t\right)}{R} \frac{\partial q\left(d_{t-1}, z_t\right)}{\partial d_{t-1}} h. \tag{F.13}$$

Combining these equations, together with $c_t = d_t + y f\left(e_t\right) - R d_{t-1}$, yields:

$$c_t^{-\gamma} = \beta R \mathbb{E}_t \left[c_{t+1}^{-\gamma} \right] + \mu_t^* - \beta \mathbb{E}_t \left[\mu_{t+1}^* \frac{\left(s + s_{t+1}\right)}{R} \frac{\partial q_{t+1}\left(d_t, z_{t+1}\right)}{\partial d_t} h \right]. \tag{F.14}$$

Now, the expression $\mu_{t+1}^* \frac{(s+s_{t+1})}{R} \frac{\partial q_{t+1}(d_t, z_{t+1})}{\partial d_t}$ can be simplified further. Recall the complementary slackness condition:

$$\mu_t^* \left[\left(s + s_t\right) q_t h / R - d_t \right] = 0. \tag{F.15}$$

Suppose the collateral constraint does not bind in $t+1$. Then, $\mu_{t+1}^* \frac{(s+s_{t+1})}{R} \frac{\partial q_{t+1}(d_t, z_{t+1})}{\partial d_t}$ simplifies to zero. Suppose, instead, the constraint binds. Then, using (F.8) and (F.9), $\frac{\partial q_{t+1}(d_t, z_{t+1})}{\partial d_t}$ simplifies to:

$$\frac{\partial q_{t+1}\left(d_t, z_{t+1}\right)}{\partial d_t} = \frac{R}{s + s_{t+1}} \left(\frac{\partial c_{t+1}}{\partial d_t} + R \right) / h. \tag{F.16}$$

As a result, the SP's Euler equation becomes:

$$c_t^{-\gamma} = \beta R \mathbb{E}_t \left[c_{t+1}^{-\gamma} \right] + \mu_t^* - \beta R \mathbb{E}_t \left[\mu_{t+1}^* \left(\frac{\partial c_{t+1}}{\partial d_t} \frac{1}{R} + 1 \right) \right]. \tag{F.17}$$

This expression can be directly compared with the corresponding relationship obtained in the DE, (F.4). It can readily be noticed that the two solutions are not equivalent to each other unless the following equality holds:

$$\mu_t = \mu_t^* - \beta R \mathbb{E}_t \left[\mu_{t+1}^* \left(\frac{\partial c_{t+1}}{\partial d_t} \frac{1}{R} + 1 \right) \right], \tag{F.18}$$

which is generally not satisfied. Thus, we need to resort to a fully numerical solution in order to compare the DE to the CEE, when considering the model with a collateral constraint based on the current-period price of the collateral asset.

F.3 Numerical implementation

We now turn to the numerical solution of the model, both in the DE and the CEE. We first present the solution methods used in each of the two cases, and then discuss the numerical results. As in the baseline model (see Appendix B), we rule out starvation points by imposing an upper bound d_{max} on the debt domain. The bound can be retrieved by combining the borrowing and budget constraints:

$$\begin{aligned} d_t &\leq (s + s_t) \frac{q_t(c_t) h}{R}, \\ c_t + R d_{t-1} - y f(e_t) &\leq (s + s_t) \frac{q_t(c_t) h}{R}, \end{aligned}$$

letting $c_t \to 0$, and solving for d_{t-1}:

$$d_{t-1} \leq \frac{y f(e_t)}{R} \quad \forall e_t \quad \Rightarrow \quad d_{t-1} \leq \min_e \frac{y f(e_t)}{R}. \tag{F.19}$$

Observe that $\lim_{c_t \to 0} q_t(c_t) = 0$. This implies that the upper bound on the debt domain is lower than in our baseline model (since, in that case, $\lim_{c_t \to 0} q_{t+1}(c_t) \neq 0$). As a result, we need to reduce the steady-state debt level in the economy, which we obtain by reducing the utility weight on durable goods, ν, by a factor of five, while keeping all the other parameters at the baseline values reported in Table 1.

Numerical solution method: Decentralized equilibrium

The solution method is based on Jeanne and Korinek (2019) and applies an endogenous grid method (EGM) that handles occasionally binding constraints. Let $\mathbf{d}_t \mathbf{z}_t^\top$ be matrices of all state combinations such that $c(\mathbf{d}_t \mathbf{z}_t^\top), q(\mathbf{d}_t \mathbf{z}_t^\top), d(\mathbf{d}_t \mathbf{z}_t^\top)$, and $\mu(\mathbf{d}_t \mathbf{z}_t^\top)$ are matrices of the same dimension. Note, the EGM implies that we use $\mathbf{d}_t$ instead of $\mathbf{d}_{t-1}$, and the upper bound of debt does not violate the starvation constraint (F.19).

1. Make initial guesses of $c^i(\mathbf{d}_t \mathbf{z}_t^\top), q^i(\mathbf{d}_t \mathbf{z}_t^\top), d^i(\mathbf{d}_t \mathbf{z}_t^\top)$, and $\mu^i(\mathbf{d}_t \mathbf{z}_t^\top)$.

2. Derive the policy functions of the *unconstrained* regime:

   (a) By definition: $\mu^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = 0$.

   (b) From (F.4):

   $$c^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = \left\{\beta R\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma}\mathbf{\Pi}^{\mathsf{T}}\right]\right\}^{\frac{-1}{\gamma}}.$$

   (c) From (F.5):

   $$q^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = \left[c^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\right]^{\gamma}\circ\left[\nu h^{-\gamma_h} + \beta\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma}\circ q^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\mathbf{\Pi}^{\mathsf{T}}\right]\right].$$

   (d) From (F.3):

   $$d^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = \frac{1}{R}\left[yf\left(\widetilde{\mathbf{e}}\right) + \widetilde{\mathbf{d}}_t - c^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\right].$$

3. Identify when the constraint binds marginally:

   $$\widetilde{\mathbf{d}}_t \geq \left(s + \widetilde{\mathbf{s}}_t\right)/R \circ q^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)h,$$

   and construct an indicator matrix, $\mathbf{X}$, with the same dimensions as $\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}$, and which equals one when the constraint binds and zero otherwise.

4. Derive the policy functions in the *constrained* regime:

   (a) From the binding constraint (F.3), we obtain:

   $$q^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = R\left(s + \widetilde{\mathbf{s}}_t\right)^{-1}\circ\widetilde{\mathbf{d}}_t.$$

   (b) From (F.5):

   $$\begin{aligned} c^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = \Big\{&\left[\left(1 - \left(s + \widetilde{\mathbf{s}}_t\right)/R\right)\circ q^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\right]^{-1} \\ &\circ\left[\nu h^{-\gamma_h} + \beta\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma}\circ q^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\mathbf{\Pi}^{\mathsf{T}}\right] \right.\\ &\left.-q^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\left(s + \widetilde{\mathbf{s}}_t\right)\beta\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma}\mathbf{\Pi}^{\mathsf{T}}\right]\right]\Big\}^{\frac{-1}{\gamma}}. \end{aligned}$$

   (c) From (F.4):

   $$\mu^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = c^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma} - \beta R\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma}\mathbf{\Pi}^{\mathsf{T}}\right].$$

   (d) Lastly, from (F.2):

   $$d^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right) = \frac{1}{R}\left[yf\left(\widetilde{\mathbf{e}}\right) + \widetilde{\mathbf{d}}_t - c^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\mathsf{T}}\right)\right].$$

5. For each combination of exogenous shock, $j = 1, .., n_z$, a threshold value of debt, $\widehat{d}_j$, that ensures a marginally binding constraint is identified through interpolation on:

$$\widetilde{\mathbf{d}}_{tj} - (s + \widetilde{\mathbf{s}}_t) / R \circ q^{uncon} \left(\mathbf{d}_t \mathbf{z}_{tj}^{\mathsf{T}} \right) h = 0.$$

The scalar, $\widehat{d}_j$, is then added to each of the policy functions:

$$\widehat{y}_j = \left[\widehat{y}_j^{unc} \left(d_j < \widehat{d}_j \right), \widehat{y}^{unc} \left(\widehat{d}_j \right), \widehat{y}^{con} \left(d_j > \widehat{d}_j \right) \right]^{\mathsf{T}}, \tag{F.20}$$

for each:

$$y_j \in \left\{ c^i \left(\mathbf{d}_t \mathbf{z}_{tj}^{\mathsf{T}} \right), q^i \left(\mathbf{d}_t \mathbf{z}_{tj}^{\mathsf{T}} \right), d^i \left(\mathbf{d}_t \mathbf{z}_{tj}^{\mathsf{T}} \right), \mu^i \left(\mathbf{d}_t \mathbf{z}_{tj}^{\mathsf{T}} \right) \right\}.$$

Note, $\mathbf{X}$ is used to determine when $d_j < \widehat{d}_j$. Then $\widehat{d}_j$ can be used to interpolate $\widehat{y}_j$ onto $\widetilde{\mathbf{d}}_{tj}$ to construct $y^{i+1} \left(\mathbf{d}_t \mathbf{z}_{tj}^{\mathsf{T}} \right)$. We then interpolate these new policy functions, $\widehat{y}_j$, as a function of $\left[\widehat{d}_j^{unc} \left(d_j < \widehat{d}_j \right), \widehat{d}_j, \widehat{d}_j^{unc} \left(d_j > \widehat{d}_j \right) \right]^{\mathsf{T}}$, onto the grid of debt tomorrow, $\widetilde{\mathbf{d}}_{tj}$, to construct $y^{i+1} \left(\mathbf{d}_t \mathbf{z}_{tj}^{\mathsf{T}} \right)$. Then merge all the policy functions together for each $j = 1, .., n_z$:

$$c^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right), q^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right), d^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right), \mu^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right),$$

while denoting the old policy functions with a superscript i.

6. To evaluate convergence the following metric is used:

$$\left\| \text{vec} \left[c^{i+1} \left(\mathbf{d}_{t-1} \mathbf{z}_t^{\mathsf{T}} \right) \right] - \text{vec} \left[c^i \left(\mathbf{d}_{t-1} \mathbf{z}_t^{\mathsf{T}} \right) \right] \right\|_\infty < \varepsilon,$$
$$\left\| \text{vec} \left[q^{i+1} \left(\mathbf{d}_{t-1} \mathbf{z}_t^{\mathsf{T}} \right) \right] - \text{vec} \left[q^i \left(\mathbf{d}_{t-1} \mathbf{z}_t^{\mathsf{T}} \right) \right] \right\|_\infty < \varepsilon.$$

If the conditions are satisfied, then stop. If not, update the policy functions according to:

$$c^{i+2} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) = \omega_c c^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) + (1 - \omega_c) c^i \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right),$$
$$q^{i+2} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) = \omega_q q^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) + (1 - \omega_q) q^i \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right),$$
$$\mu^{i+2} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) = \omega_\mu \mu^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) + (1 - \omega_\mu) \mu^i \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right),$$
$$d^{i+2} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) = \omega_d d^{i+1} \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right) + (1 - \omega_d) d^i \left(\mathbf{d}_t \mathbf{z}_t^{\mathsf{T}} \right),$$

where ω_y are weights. Reset $i + 2$ to i and return to Step 2.

Numerical solution method: Constrained-efficient equilibrium

We now turn to describe the solution method employed to solve the model in the CEE. The method proceeds in parallel to the one employed for the DE, with the main difference being that we also need to solve for the policy function of μ_t^* (i.e., the Lagrange multiplier associated with the collateral constraint from the perspective of the SP), in this case.

1. Make initial guesses of $c^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right), q^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right), d^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)$, $\mu^{*i}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)$, and $\mu^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)$ using the policy functions of the decentralized equilibrium.

2. Derive the policy functions in the *unconstrained* regime:

   (a) By definition: $\mu^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) = \mu^*\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) = 0$.

   (b) From (F.17):

$$c^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) = \left\{\beta R\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)^{-\gamma} - \mu^{*i}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)\circ\left[\frac{\partial c^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)}{\partial d_t}/R+1\right]\right]\mathbf{\Pi}^{\intercal}\right\}^{\frac{-1}{\gamma}}.$$

   The derivative, $\partial c^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)/\partial d_t$, is obtained by applying a central-finite difference scheme. Specifically:

$$\left[\partial c^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)/\partial d_t\right]_{i_d,i_z} \approx \frac{\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)\right]_{i_d+1,i_z} - \left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)\right]_{i_d-1,i_z}}{\left[\widetilde{\mathbf{d}}_t\right]_{i_d+1,i_z} - \left[\widetilde{\mathbf{d}}_t\right]_{i_d-1,i_z}}.$$

   Here, i_d denotes the index in the debt dimension, and i_z is the index in the exogenous shock dimension of the matrices.

   (c) From (F.10):

$$q^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) = c^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)^{\gamma}\left[\nu h^{-\gamma_h} + \beta\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)^{-\gamma}\circ q^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)\mathbf{\Pi}^{\intercal}\right]\right].$$

   (d) From (F.8):

$$d^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) = \frac{1}{R}\left[yf\left(\widetilde{\mathbf{e}}\right) + \widetilde{\mathbf{d}}_t - c^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)\right].$$

3. Identify when the constraint binds marginally:

$$\widetilde{\mathbf{d}}_t \geq \left(s+\widetilde{\mathbf{s}}_t\right)/R\circ q^{uncon}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)h,$$

   and construct an indicator matrix, $\mathbf{X}$, with the same dimensions as $\mathbf{d}_t\mathbf{z}_t^{\intercal}$, and which equals one when the constraint binds and zero otherwise.

4. Derive the policy functions of the *constrained* regime:

   (a) From the binding constraint (F.15), we obtain:

$$q^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) = R\left(s+\widetilde{\mathbf{s}}_t\right)^{-1}\circ\widetilde{\mathbf{d}}_t.$$

(b) From (F.10):

$$c^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)=\left\{\left[q^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)\circ\left[1-\left(s+\widetilde{\mathbf{s}}_t\right)/R\right]\right]^{-1}\circ\left[\nu h^{-\gamma_h}+\beta\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)^{-\gamma}\circ q^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)\mathbf{\Pi}^{\intercal}\right]\right.\right.$$
$$\left.\left.-\beta\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)^{-\gamma}\mathbf{\Pi}^{\intercal}\right]\circ q^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)\circ\left(s+\widetilde{\mathbf{s}}_t\right)\right]\right\}^{\frac{-1}{\gamma}}.$$

(c) From (F.11):

$$\mu^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)=c^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)^{-\gamma}-\beta R\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)^{-\gamma}\mathbf{\Pi}^{\intercal}\right].$$

(d) From (F.17):

$$\mu^{*con}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)=c^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)^{-\gamma}-R\beta\left[c^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)^{-\gamma}-\mu^{*i}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)\circ\left[\frac{\partial c^i\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)}{\partial d_t}/R+1\right]\right]\mathbf{\Pi}^{\intercal}.$$

(e) Lastly, from (F.8):

$$d^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)=\frac{1}{R}\left[yf\left(\widetilde{\mathbf{e}}\right)+\widetilde{\mathbf{d}}_t-c^{con}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right)\right].$$

5. For each combination of exogenous shocks, $j=1,..,n_z$, a threshold value of debt, $\widehat{d}_j$, that ensures a marginally binding constraint is identified through interpolation on:

$$\widetilde{\mathbf{d}}_{tj}-\left(s+\widetilde{\mathbf{s}}_t\right)/R\circ q^{uncon}\left(\mathbf{d}_t\mathbf{z}_{tj}^{\intercal}\right)h=0.$$

The scalar, $\widehat{d}_j$, is then added to each of the policy functions:

$$\widehat{y}_j=\left[\widehat{y}_j^{unc}\left(d_j<\widehat{d}_j\right),\widehat{y}^{unc}\left(\widehat{d}_j\right),\widehat{y}^{con}\left(d_j>\widehat{d}_j\right)\right]^{\intercal}, \tag{F.21}$$

for each:

$$y_j\in\left\{c^i\left(\mathbf{d}_t\mathbf{z}_{tj}^{\intercal}\right),q^i\left(\mathbf{d}_t\mathbf{z}_{tj}^{\intercal}\right),d^i\left(\mathbf{d}_t\mathbf{z}_{tj}^{\intercal}\right),\mu^{*i}\left(\mathbf{d}_t\mathbf{z}_{tj}^{\intercal}\right),\mu^i\left(\mathbf{d}_t\mathbf{z}_{tj}^{\intercal}\right)\right\}.$$

Note, $\mathbf{X}$ is used to determine when $d_j<\widehat{d}_j$. Then $\widehat{d}_j$ can be used to interpolate $\widehat{y}_j$ onto $\widetilde{\mathbf{d}}_{tj}$ to construct $y^{i+1}\left(\mathbf{d}_t\mathbf{z}_{tj}^{\intercal}\right)$. We then interpolate these new policy functions, $\widehat{y}_j$, as a function of $\left[\widehat{d}_j^{unc}\left(d_j<\widehat{d}_j\right),\widehat{d}_j,\widehat{d}_j^{unc}\left(d_j>\widehat{d}_j\right)\right]^{\intercal}$, onto the grid of debt tomorrow, $\widetilde{\mathbf{d}}_{tj}$, to construct $y^{i+1}\left(\mathbf{d}_t\mathbf{z}_{tj}^{\intercal}\right)$. Then merge all the policy functions together for each $j=1,..,n_z$:

$$c^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right),q^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right),d^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right),\mu^{*i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right),\mu^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right).$$

6. To evaluate convergence the following metric is used:

$$\left\|\text{vec}\left[c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\intercal}\right)\right] - \text{vec}\left[c^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\intercal}\right)\right]\right\|_{\infty} < \varepsilon,$$
$$\left\|\text{vec}\left[q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\intercal}\right)\right] - \text{vec}\left[q^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\intercal}\right)\right]\right\|_{\infty} < \varepsilon.$$

If the conditions are satisfied, then stop. If not, update the policy functions according to:

$$\begin{aligned}
c^{i+2}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) &= \omega_c c^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) + (1-\omega_c)\, c^{i}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right),\\
q^{i+2}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) &= \omega_q q^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) + (1-\omega_q)\, q^{i}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right),\\
\mu^{*i+2}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) &= \omega_{\mu^*} \mu^{*i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) + (1-\omega_{\mu^*})\, \mu^{*i}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right),\\
\mu^{i+2}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) &= \omega_\mu \mu^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) + (1-\omega_\mu)\, \mu^{i}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right),\\
d^{i+2}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) &= \omega_d d^{i+1}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right) + (1-\omega_d)\, d^{i}\left(\mathbf{d}_t\mathbf{z}_t^{\intercal}\right),
\end{aligned}$$

where ω_y are weights. Reset $i+2$ to i and return to Step 2. Note that the resulting policy functions are functions of d_{t-1} and z_t.

F.4 Numerical evidence

We are now ready to present the numerical results. We begin by considering the unconditional welfare loss at our baseline calibration, in each of the two model equilibria. In the DE, λ amounts to -0.0162%. In the CEE, an even larger welfare gain of -0.0464% was obtained. Thus, in both cases, business cycles are beneficial to household welfare, relative to the deterministic scenario.

We report the policy functions in Figure F.1 (DE) and Figure F.2 (CEE). These are relatively similar across the two cases, though not identical. It may be interesting to compare the policy functions obtained with a current-price collateral constraint to those obtained in the case of a future-price collateral constraint, as reported in Figure E.2. The most significant difference is observed for current debt, d_t, as a function of past debt, as we also remark in Section 6.3.

On the role of different shocks

In Figure F.3, we conduct an exercise similar to the one reported in Figure 3, this time considering the model with a current-period collateral constraint. We begin by considering the first row, which refers to the DE case. From the left panel, we observe that—conditional on no credit-limit shocks—larger income shocks raise the welfare *cost* of business cycles monotonically. Thus, unlike the model with an expected-future price of the collateral asset, there is no range of values of σ_e for which the endogenous switching effect dominates. In the right panel, we see that larger credit-limit shocks always have the effect of increasing the welfare *gain* from uncertainty, exactly as observed in Figure 3. In the second row, we plot the results of the corresponding exercise for the CEE case. The main takeaway is that the overall message from the top row is confirmed: larger income shocks make business cycles more costly, all else equal, while larger credit shocks have the opposite effect.[21]

Finally, the bottom row of Figure F.3 reports the welfare gap between the DE and the CEE. The left panel makes it clear that the SP obtains a smaller welfare loss than the DE, for all possible values of the standard deviation of the income process, conditional on no credit-limit shocks. In this case, the DE entails *overborrowing* relative to the constrained-efficient case. In the CEE, the SP internalizes the pecuniary externality at play in the DE, thereby reducing debt and paving the way for higher consumption.[22] The right panel of the figure shows that, for small to moderate magnitudes of the credit-limit shock, and conditional on no income shocks, the SP obtains a larger welfare gain than observed in the DE. However, for very large credit-limit shocks, the CEE does *worse* than the DE, from a welfare viewpoint. To understand this result, we find it useful to consider the first-order condition for the SP's choice of debt (F.14), which we repeat here for convenience:

$$c_t^{-\gamma} = \beta R \mathbb{E}_t \left[c_{t+1}^{-\gamma} \right] + \mu_t^* - \beta \mathbb{E}_t \left[\mu_{t+1}^* \frac{(s + s_{t+1})}{R} \frac{\partial q_{t+1}(d_t, z_{t+1})}{\partial d_t} h \right].$$

As seen from this equation, the expected-future credit-limit shock s_{t+1} appears explicitly in the last term on the right-hand side, reflecting that the SP takes care of the pecuniary externality. Thus, when credit-limit shocks become sufficiently large, such internalization "fires back", as it induces the SP to reduce debt abruptly, in the case of a negative shock realization. This exacerbates the risk of inducing a debt-deflation spiral, which is a characteristic trait of this model configuration, as compared with the one embedding the expected-future price of the collateral asset, where such episodes are not possible, as discussed in Section 6.3.

[21] Notice that λ turns negative for relatively small income shocks. As the variance of the shocks increases from zero, the term that internalizes the pecuniary externality of the SP's Euler equation (F.14), $-\frac{\beta h s}{R} \mathbb{E}_t \left[\mu_{t+1}^* \frac{\partial q_{t+1}(d_t, e_{t+1})}{\partial d_t} \right]$, increases. This results in a reduction of debt and an increase in consumption. See Juul (2023) for further details on this mechanism.

[22] In the DE economy, the mean level of debt is 0.5143, while mean consumption is 0.9948. In the CEE economy, instead, the corresponding level of debt is lower (0.4971), while average consumption is slightly higher (0.9950).

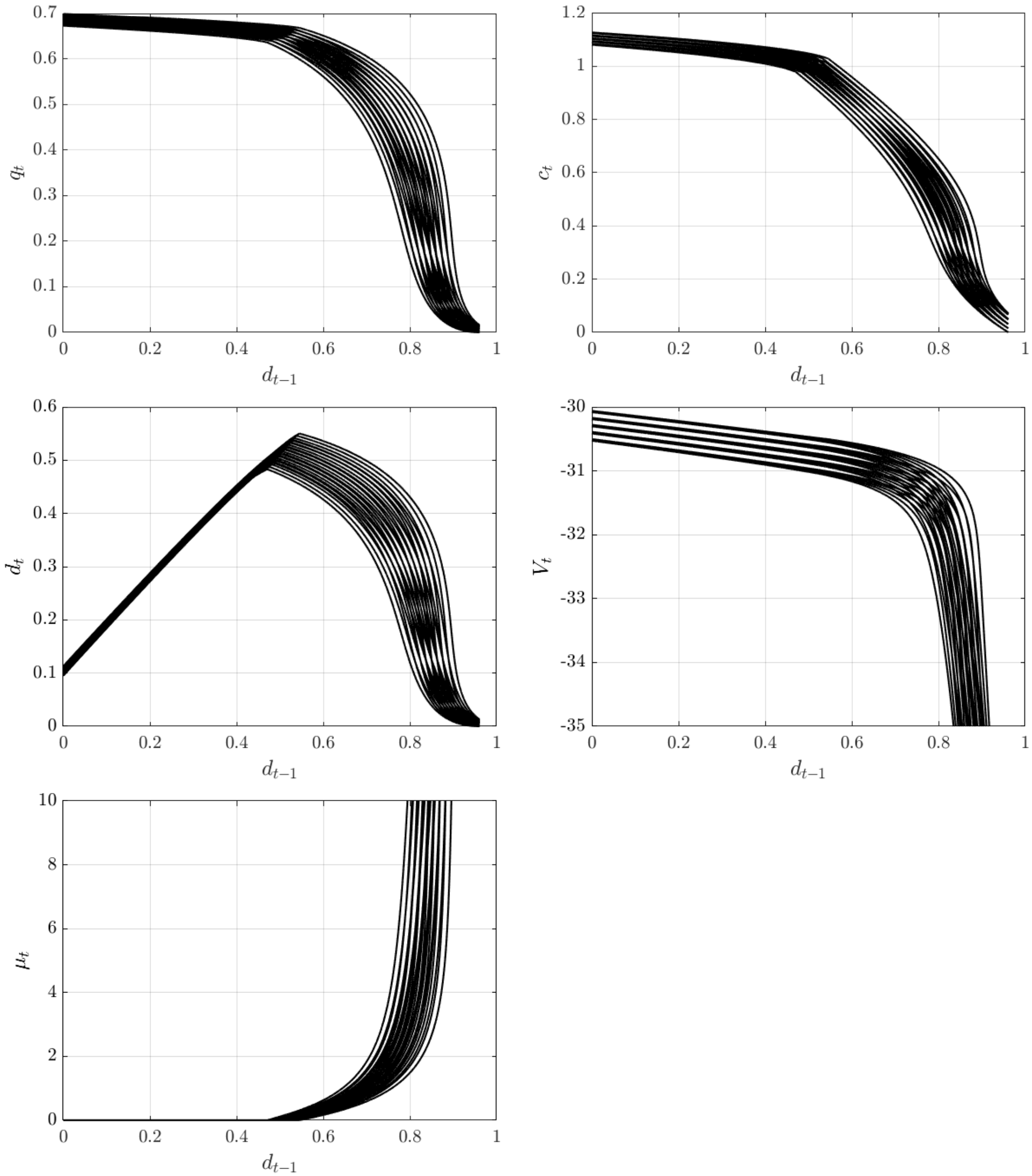

Figure F.1: Policy functions in the DE of the q_t-economy.

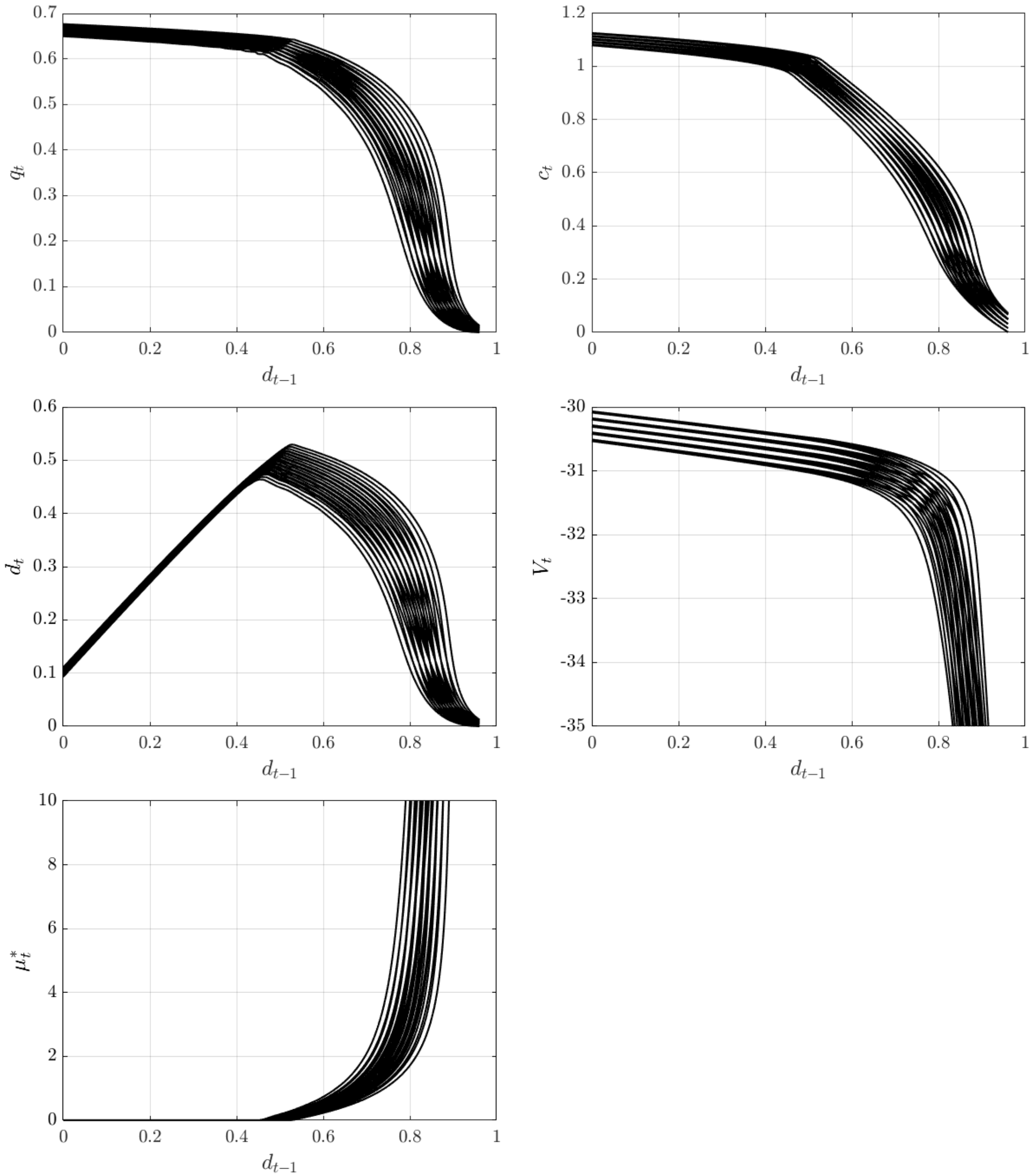

Figure F.2: Policy functions in the CEE of the q_t-economy.

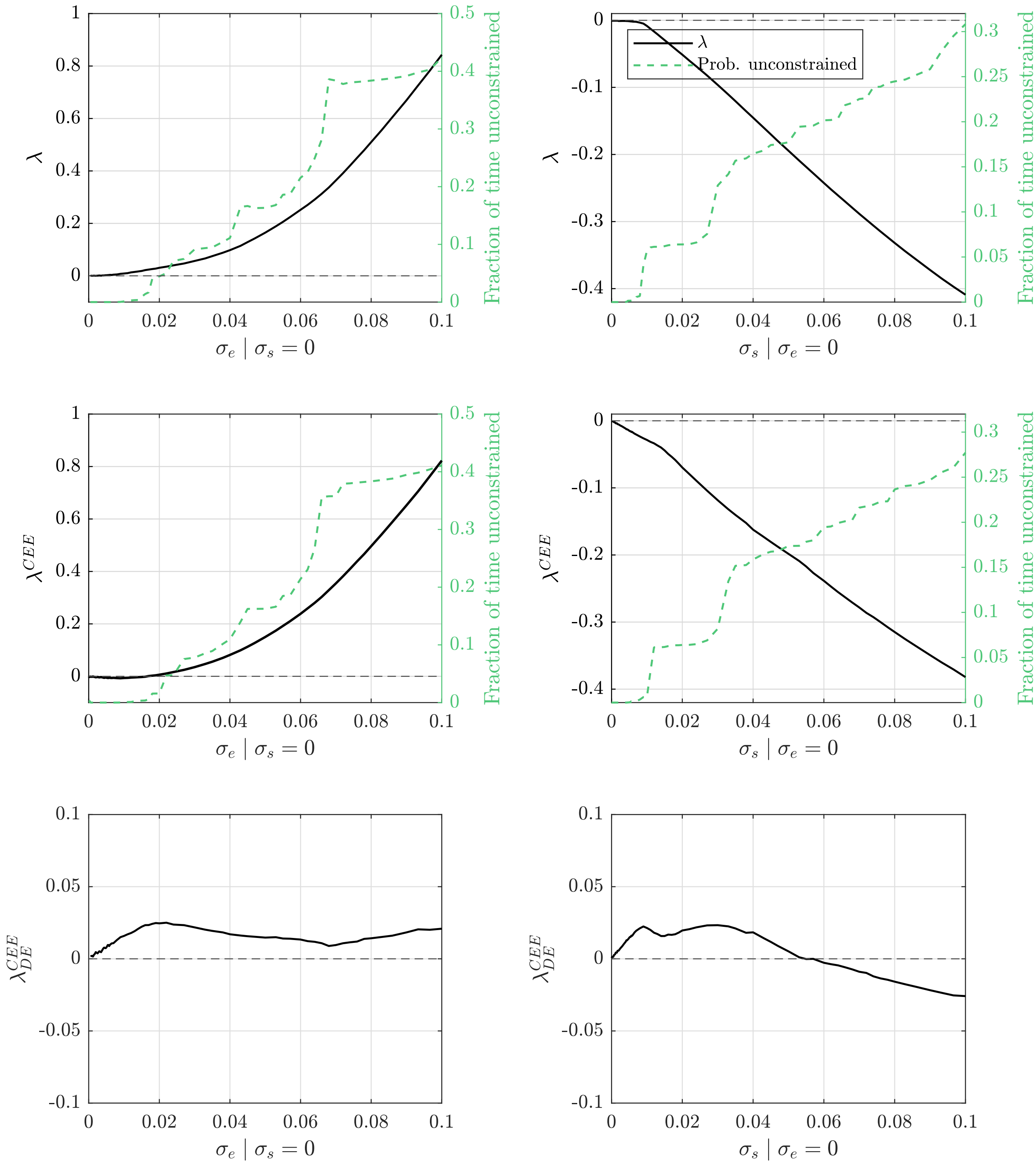

Figure F.3: Varying uncertainty and welfare in the q_t-economy. In each panel, the continuous (black) line reports a welfare measure/comparison for different standard deviations of a given shock, conditional on the other shock being switched off (all the other parameters are at their baseline values). Specifically, rows 1 and 2 compute λ in the DE and the CEE equilibrium, respectively. Row 3, instead, compares the value function in the DE with that in the CEE (values above zero indicate that the CEE case yields a welfare *gain* relative to the DE case). The dashed (green) line indicates the frequency of episodes in which the financial constraint is slack.

G Fixed credit limits

We now return to the case of a fixed credit limit considered in Section 6.4, i.e., where the borrowing constraint (4) is replaced by $d_t \leq (s+s_t)\frac{qh}{R}$. In Figure G.1, we examine how the computation of λ changes with the magnitude of each of the two shocks. As seen from the left panel, the presence of exogenous perturbations of the credit limit is now crucial for the emergence of welfare gains: When we shut off this shock so that the borrowing limit is completely fixed, business cycles become costly. This is consistent with the findings of İmrohoroğlu (1989) in a model with an Aiyagari (1994)-style constant borrowing limit, which by construction does not embody the financial accelerator. In the absence of credit-limit shocks, the endogenous switching effect has little traction, since very large income shocks are required to present households with the perspective of being unconstrained. By contrast, the right panel of the figure indicates that, conditional on no income shocks, raising the standard deviation of financial shocks leads to an increasing welfare gain, much like in our baseline model (see Figure 3), although the gain is smaller, in this case.

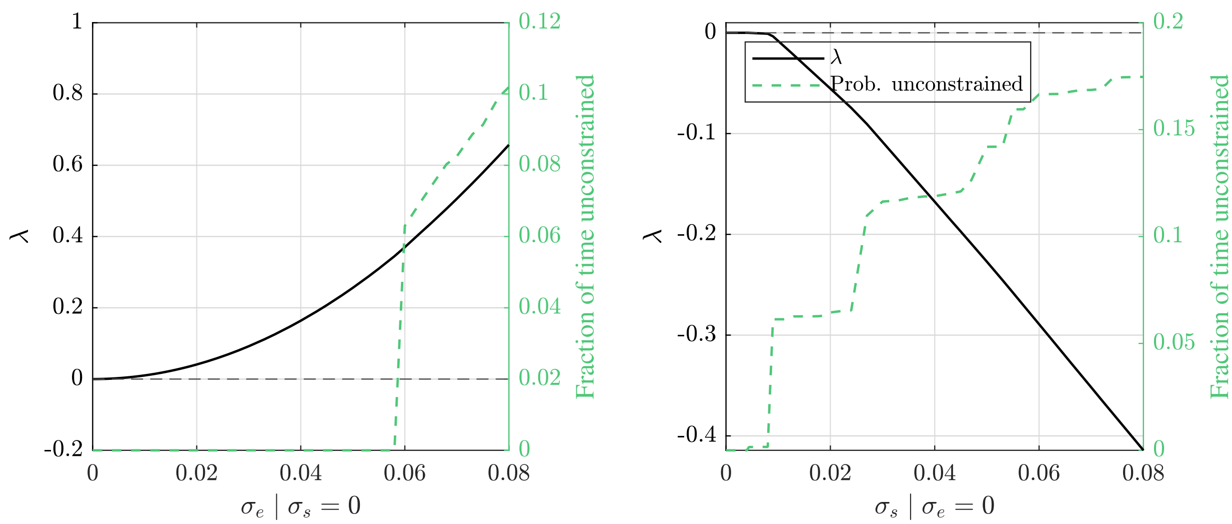

Figure G.1: Varying uncertainty in the economy with fixed credit limits. In each panel, the continuous (black) line reports λ for different standard deviations of a given shock, conditional on the other shock being switched off. The dashed (green) line indicates the frequency of episodes in which the financial constraint is slack. All the other parameters are at their baseline values.

H Partial equilibrium

To devise a partial equilibrium economy, we abstain from imposing the market supply of durable goods (11). As a result, the demand for durable goods will now vary through (10), while the relative price is assumed to be exogenous, that is, $q_t = \overline{q}$. In this setting, households optimize with respect to nondurable goods, c_t, durable goods, h_t, and debt, d_t, conditional on the debt level of the last period and the stock of durables, d_{t-1} and h_{t-1}, as well as the realized income and loan-to-value shocks, e_t and s_t. To solve for partial equilibrium, we employ a value-function iteration procedure, following Guerrieri and Iacoviello (2017). As the computational problem increases in complexity, due to the introduction of a new state variable, h_{t-1}, we need to adjust the calibration to ensure an accurate global solution. This leads us to choose a lower value of ν, in accordance with Section F.3, thus reducing the size of the debt domain.

H.1 Policy functions

Here, we present figures of the policy functions of the partial equilibrium for different levels of the relative price—low ($0.9 \times q$), medium ($1.0 \times q$), and high ($1.1 \times q$)—and for different levels of standard deviations of the shocks—(σ_e, σ_s), $(\sigma_e = 0, \sigma_s)$, and $(\sigma_e, \sigma_s = 0)$. These are reported in Figure H.1. For comparison, the decentralized general equilibrium model's policy functions are plotted as well. To enhance comparability, households in partial equilibrium have a unitary amount of durable goods.

H.2 Numerical evidence

We then turn to the numerical evidence generated by the partial equilibrium model. Figure H.2 reports the results of an exercise in which we gradually raise the standard deviation of one shock at a time, while shutting off the other shock entirely. The left panel of the figure clearly shows that conditional on no credit-limit shocks, larger income shocks are associated with an increasingly large welfare gain. In contrast, as seen from the right panel, credit-limit shocks in isolation lead to a welfare loss from uncertainty.

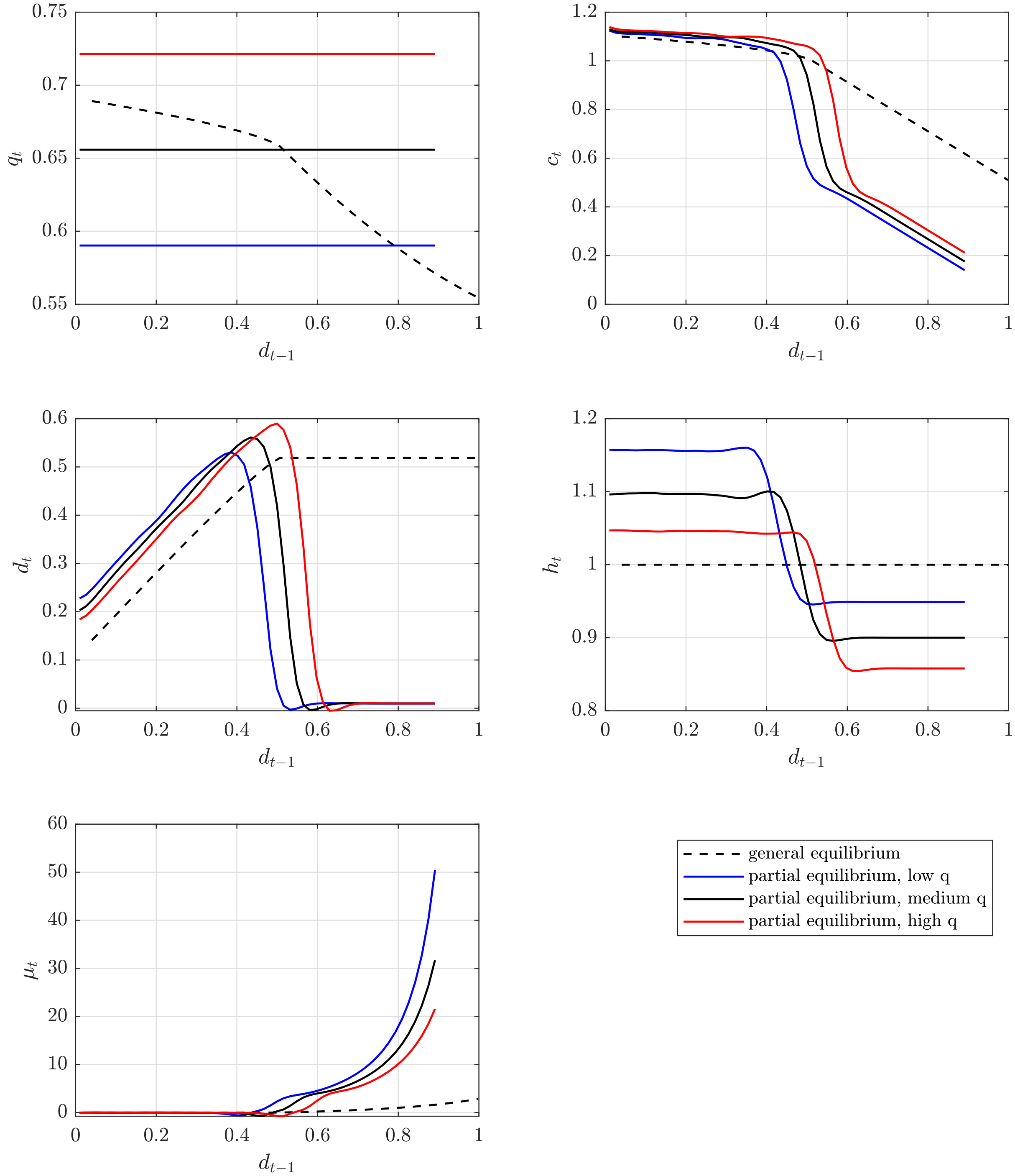

Figure H.1: Policy functions of the partial-equilibrium (for various price levels) and the general-equilibrium settings for the baseline calibration of shock volatilities.

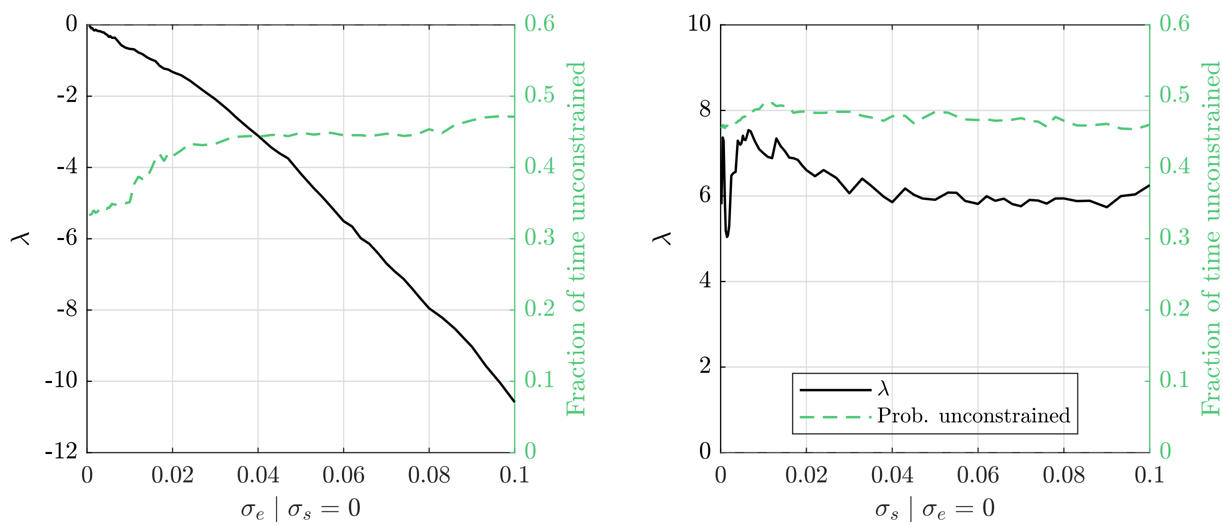

Figure H.2: Welfare costs of business cycles in partial equilibrium. We consider different standard deviations of a given shock, conditional on the other shock being switched off. All the other parameters are set at their baseline values.

I Nonbinding collateral constraint in the steady state

We now consider a variation of the model where the deterministic steady state features a nonbinding collateral constraint. To ensure stationarity, we follow Schmitt-Grohé and Uribe (2003) and impose convex portfolio adjustment costs on borrowing abroad. The resulting deterministic steady state is efficient, as the borrowing constraint is nonbinding. Aside from this difference, the model is similar to the baseline version considered in Section 2.

I.1 Model environment

The household maximizes (1) by choosing d_t and h_t subject to (4) and:

$$yf\left(e_t\right) - Rd_{t-1} = q_t\left(h_t - h_{t-1}\right) - d_t + c_t + \psi\left(d_t - \bar{d}\right)^2/2. \tag{I.1}$$

The term $\psi\left(d_t - \overline{d}\right)^2/2$ is a convex adjustment cost of foreign debt, which the household must pay whenever debt deviates from its steady-state level $\bar{d}$. The parameter ψ governs the magnitude of this cost.

The Lagrangian reads:

$$\begin{aligned}\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t & \left[\left(c_t^{1-\gamma}/(1-\gamma) + \nu h_t^{1-\gamma_h}/(1-\gamma_h)\right)\right. \\ & + \lambda_t\left(yf\left(e_t\right) - Rd_{t-1} - q_t\left(h_t - h_{t-1}\right) + d_t - c_t - \psi\left(d_t - \overline{d}\right)^2/2\right) \\ & \left.+ \mu_t\left(\left(s + s_t\right)\mathbb{E}_t\left[q_{t+1}\right]h_t/R - d_t\right)\right].\end{aligned} \tag{I.2}$$

Collapsing the FOCs and fixing the supply of durables, i.e. $h_t = h$, yield the following set of equilibrium equations:

$$c_t - d_t = yf\left(e_t\right) - Rd_{t-1} - \psi\left(d_t - \bar{d}\right)^2/2, \tag{I.3}$$

$$\nu h^{-\gamma_h} = c_t^{-\gamma} q_t - \mu_t\left(s + s_t\right)\frac{\mathbb{E}_t\left[q_{t+1}\right]}{R} - \beta\mathbb{E}\left\{c_{t+1}^{-\gamma} q_{t+1}\right\}, \tag{I.4}$$

$$c_t^{-\gamma}\left[1 - \psi\left(d_t - \bar{d}\right)\right] = \mu_t + R\beta\mathbb{E}\left\{c_{t+1}^{-\gamma}\right\}, \tag{I.5}$$

$$d_t \leq \left(s + s_t\right)\frac{\mathbb{E}_t\left[q_{t+1}\right]h_t}{R}. \tag{I.6}$$

Steady state and calibration

In the absence of stochastic shocks, we assume that the credit constraint is nonbinding, i.e., that (I.6) holds with a strict inequality. By complementary slackness, this implies that $\mu = 0$. The steady-state version of (I.5) then boils down to $\beta = \frac{1}{R}$, given that $d = \bar{d}$. This leaves us with the steady-state versions of (I.3) and (I.4), which can be written as:

$$c = y + \bar{d}(1 - R), \tag{I.7}$$

$$q = \frac{\nu h^{-\gamma_h}}{\left(1 - \beta\right)c^{-\gamma}}. \tag{I.8}$$

To make the comparison with our baseline model as clean as possible, we set $\bar{d}$ so as to ensure a steady-state ratio of household debt to annual GDP of 0.63, as described in Section 4. This allows us to obtain steady-state consumption from (I.7) and, in turn, the asset price from (I.8) after calibrating ν to match the desired skewness of consumption growth of -0.9, as in the baseline model. This implies a value of $\nu = 0.035$. With the exception of β and ν, all other parameters are identical to those used in our baseline model (see Table 1). Finally, the parameter that governs the portfolio adjustment cost, ψ, is set to the lowest value that ensures stationarity, which is $\psi = 0.0042$.[23]

I.2 Numerical implementation

We adjust the solution algorithm in Appendix B to a model with an unconstrained steady state. We generate matrices encompassing all possible combinations of states, represented as $\mathbf{d}_{t-1}\mathbf{z}t^{\mathsf{T}}$, and then aim to derive the subsequent policy functions in matrix structure: $c\left(\mathbf{d_{t-1}z_t}^{\mathsf{T}}\right)$, $q\left(\mathbf{d_{t-1}z_t}^{\mathsf{T}}\right)$, $d\left(\mathbf{d_{t-1}z_t}^{\mathsf{T}}\right)$, and $\mu\left(\mathbf{d_{t-1}z_t}^{\mathsf{T}}\right)$. These four policy functions must satisfy the four equations: (I.3)-(I.6). The solution method proceeds in the following steps:

1. Generate a discrete grid of the state space and use the steady state values as initial values for the policy functions $c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$, $q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$, $d^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$, and $\mu^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$.

2. Use (I.3) to solve for debt:

   $$d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = [2\mathbb{A}]^{-1} \circ \left[-\mathbb{B} \pm \left(\mathbb{B}^2 - 4\mathbb{A} \circ \mathbb{C}\right)^{\frac{1}{2}}\right],$$

   where:

   $$\mathbb{A} = \frac{\psi}{2},$$
   $$\mathbb{B} = -\left(1 + \psi\bar{d}\right),$$
   $$\mathbb{C} = \frac{\psi}{2}\bar{d}^2 + c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) + R\widetilde{\mathbf{d}}_{t-1} - yf\left(\widetilde{\mathbf{e}}_t\right).$$

   Note that we choose the root that is consistent with non-explosive dynamics.

3. Compute future values by applying $\mathbf{d}_t = d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$ to interpolate on $c^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$ and $q^i\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)$ to obtain $\mathbf{c}_{t+1} = \widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)$ and $\mathbf{q}_{t+1} = \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)$.

4. Derive the policy functions from the unconstrained regime:

   (a) Using the assumption of being unconstrained implies that $\mu^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \mathbf{0}$.

   (b) From (I.5), we obtain consumption:

   $$c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \left\{\left[\left[1 - \psi\left(d\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) - \bar{d}\right)\right]\right]^{-1} \circ R\beta\left[\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma}\Pi^{\mathsf{T}}\right]\right\}^{\frac{-1}{\gamma}},$$

[23]Coincidentally, we obtain a quarterly frequency of ´Sudden Stop' episodes—i.e., the constraint binds, and net capital outflows exceed one standard deviation, as defined in Bianchi (2011)—of 1.39%. This falls very close to the empirical counterpart of 1.38% in Eichengreen *et al.* (2006).

where, as in Appendix B, $\Pi \equiv \mathbf{P_e} \otimes \mathbf{P_s}$ denotes the transition matrix for $\mathbf{z}_t$.

(c) From (I.4) we recover the unconstrained asset price:

$$q^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{\gamma}\left\{\nu h^{-\gamma_h} + \beta\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma} \circ \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}\right\}.$$

(d) Utilizing the unconstrained quantities, we obtain, from (I.3), the unconstrained debt policy function:

$$d^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = [2\mathbb{A}]^{-1} \circ \left[-\mathbb{B} \pm \left(\mathbb{B}^2 - 4\mathbb{A} \circ \mathbb{C}^{uncon}\right)^{\frac{1}{2}}\right],$$

where $\mathbb{C}^{uncon} = \frac{\psi}{2}\bar{d}^2 + c^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) + R\widetilde{\mathbf{d}}_{t-1} - yf\left(\widetilde{\mathbf{e}}_t\right)$.

5. Derive policy functions from the constrained regime:

(a) Determine the restricted regime. The subsequent inequality pinpoints the conditions within $\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}$ in which the constraint becomes effective:

$$d^{uncon}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) > \left(s + \widetilde{\mathbf{s}}\right)h/R \circ \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}.$$

Based on the inequality, an identifier is constructed such that for any matrix, $\mathbf{X}_t$, denote by $[\mathbf{X}_t]^j$ the j^{th} column of $\mathbf{X}_t$ only consisting of such identified states.

(b) From (I.6) constrained debt is obtained:

$$\left[d^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \left[\left(s + \widetilde{\mathbf{s}}\right)h/R \circ \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}\right]^j, \quad \forall j.$$

(c) From (I.3) constrained consumption is:

$$\left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \left[d^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) - \psi\left[d^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) - \bar{d}\right]^2/2 - R\widetilde{\mathbf{d}}_{t-1} + yf\left(\widetilde{\mathbf{e}}_t\right)\right]^j, \quad \forall j.$$

(d) From (I.5) the constrained multiplier is given as:

$$\left[\mu^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \left[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{-\gamma} \circ \left[1 - \psi\left(d_t - \bar{d}\right)\right] - R\beta\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma}\Pi^{\mathsf{T}}\right]^j, \quad \forall j.$$

(e) From (I.4) the constrained relative price of assets is obtained:

$$\begin{aligned}\left[q^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]^j = \Bigg[c^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)^{\gamma} \circ \Bigg\{&\nu h^{-\gamma_h} \\ &+ \mu^{con}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) \circ (s + s_t) \circ \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}/R \\ &+\beta\widehat{c}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)^{-\gamma} \circ \widehat{q}\left(\mathbf{d}_{t-1}\mathbf{z}_{t+1}^{\mathsf{T}}\right)\Pi^{\mathsf{T}}\Bigg\}\Bigg]^j, \quad \forall j.\end{aligned}$$

6. From both the unconstrained and constrained regimes, we formulate a new collection of

policy functions using the identifier:

$$c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), d^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right), \mu^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right).$$

7. Convergence depends on the following metrics:

$$\left\|\text{vec}\left[c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right] - \text{vec}\left[c^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]\right\|_{\infty} < \varepsilon,$$
$$\left\|\text{vec}\left[q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right] - \text{vec}\left[q^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right)\right]\right\|_{\infty} < \varepsilon.$$

Here, ε is a tolerance criterion. If the conditions are satisfied, then stop. If not, update the policy functions according to:

$$c^{i+2}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \omega_c c^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) + (1-\omega_c)\, c^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right),$$
$$q^{i+2}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) = \omega_q q^{i+1}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right) + (1-\omega_q)\, q^{i}\left(\mathbf{d}_{t-1}\mathbf{z}_t^{\mathsf{T}}\right),$$

where ω_c and ω_q are weights. Reset $i+2$ to i and return to Step 2.

I.3 Numerical evidence

Figure I.1 reports the policy functions. In qualitative terms, these are relatively similar to those obtained from our baseline model with a financially constrained steady state (see Figure E.2 in Appendix E.4). Most importantly, the kink in consumption and debt determination occurs in both settings.

We obtain $\lambda = -0.0432\%$ under the calibration presented in Section I.1. In other words, business cycles are associated with a gain in this model environment, although smaller than in our baseline model. To shed further light on the drivers of this gain, we find it useful to consider Figure I.2, which shows how λ evolves as we raise the standard deviation of either shock, conditional on the other shock being shut off, as well as the associated frequency with which the credit constraint is *nonbinding*. Consider first that only income shocks perturb the economy (left panel). When income shocks are very small, the fluctuations effect determines a welfare loss, albeit a small one. As shocks become larger, the risk of switching to a constrained regime becomes conspicuous. This induces households to increase precautionary savings with the exact aim of avoiding such situations. As a result, they are able to remain unconstrained more than 90 percent of the time, even in the face of large shocks. This requires ever larger savings, which translate into lower average debt and higher average consumption, as seen from the left panel of Figure I.3. This paves the way for welfare gains to emerge over a substantial portion of the support for σ_e. Finally, when income shocks become extremely large, negative shock realizations are associated with rare, yet painful, reductions in consumption, so that uncertainty eventually becomes costly.

We then turn to the case of credit-limit shocks (right panel). When these are small, λ revolves around zero. Since fluctuations take place in a neighborhood of the steady state, household debt is very unlikely to reach the borrowing limit, even in the face of negative shocks. As a result of this, credit-limit shocks have little grip: Agents are not harmed by fluctuations in their borrowing capacity, nor can they exploit them. As shocks become larger, endogenous switching gains traction, and households take advantage of uncertainty to obtain a welfare gain from fluctuations by

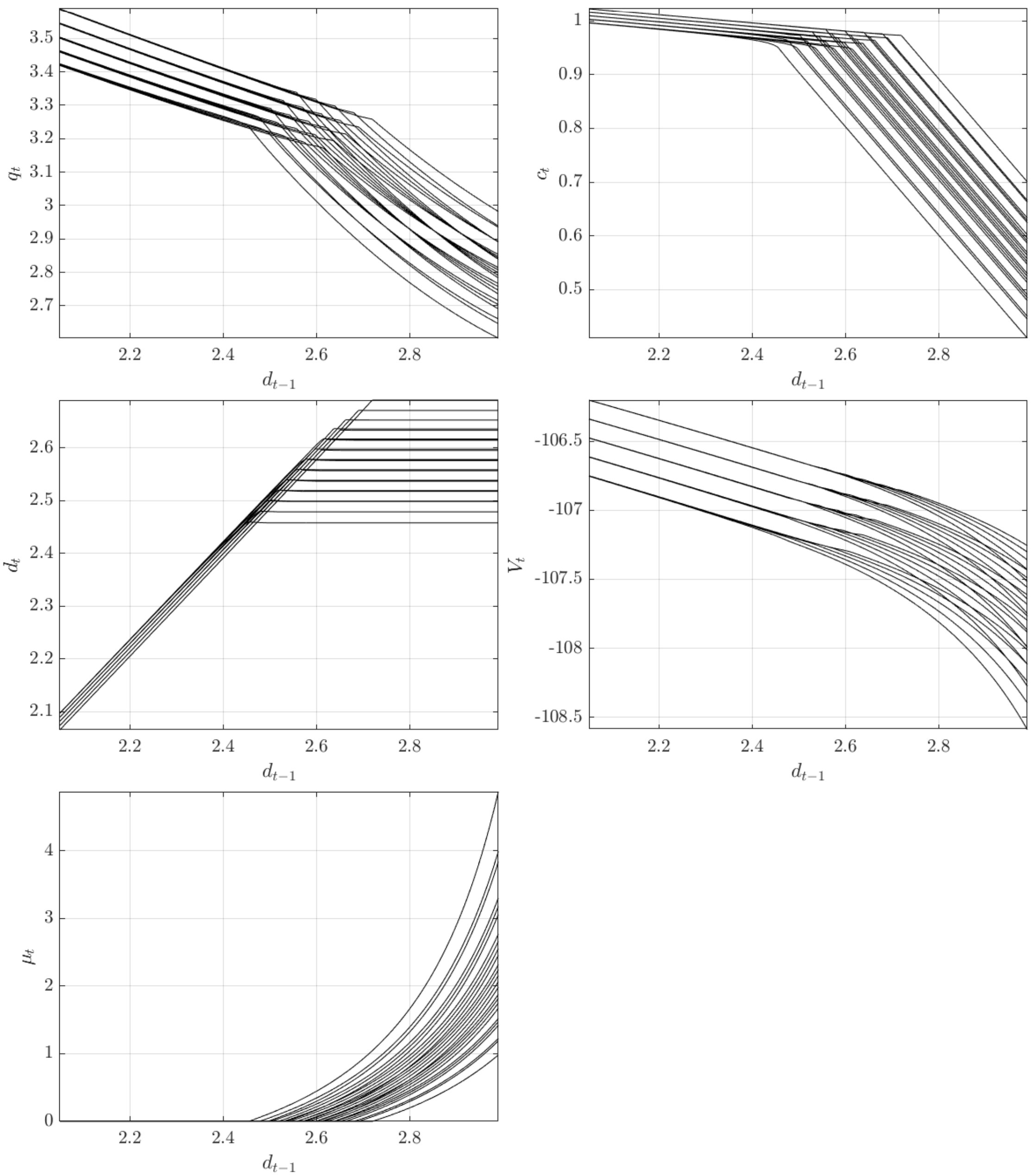

Figure I.1: Policy functions for the model with an unconstrained steady state.

raising their precautionary savings to avoid switching to a constrained regime, thus paving the way for an increase in average consumption (see Figure I.3, right panel).

Figure I.2 also allows us to shed light on the smaller welfare gain obtained in this case, compared to the baseline model. When the volatility of credit-limit shocks is moderate—i.e., below or close to our calibrated value—such shocks are rather ineffective in triggering endogenous switching. Effectively, only income shocks 'contribute' to the welfare gain. Furthermore, portfolio adjustment costs necessarily subtract resources from consumption whenever the economy is outside the steady state.

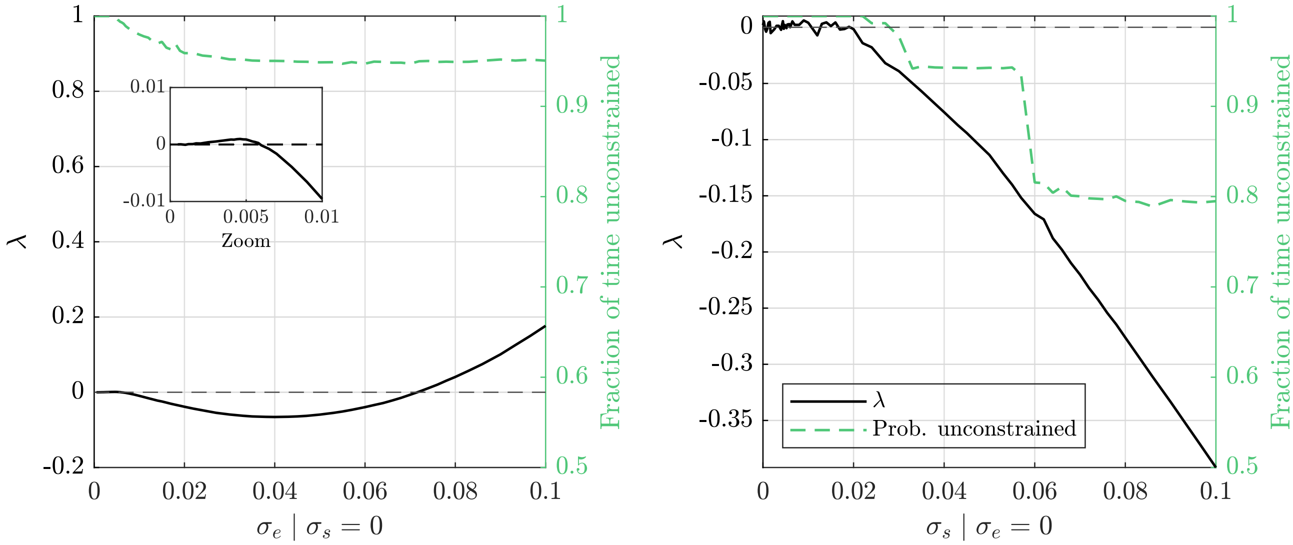

Figure I.2: Varying uncertainty in the economy with a nonbinding steady state. In each panel, the continuous (black) line reports λ for different standard deviations of a given shock, conditional on the other shock being switched off. The dashed (green) line indicates the frequency of episodes in which the financial constraint is slack. All the other parameters are at their baseline values.

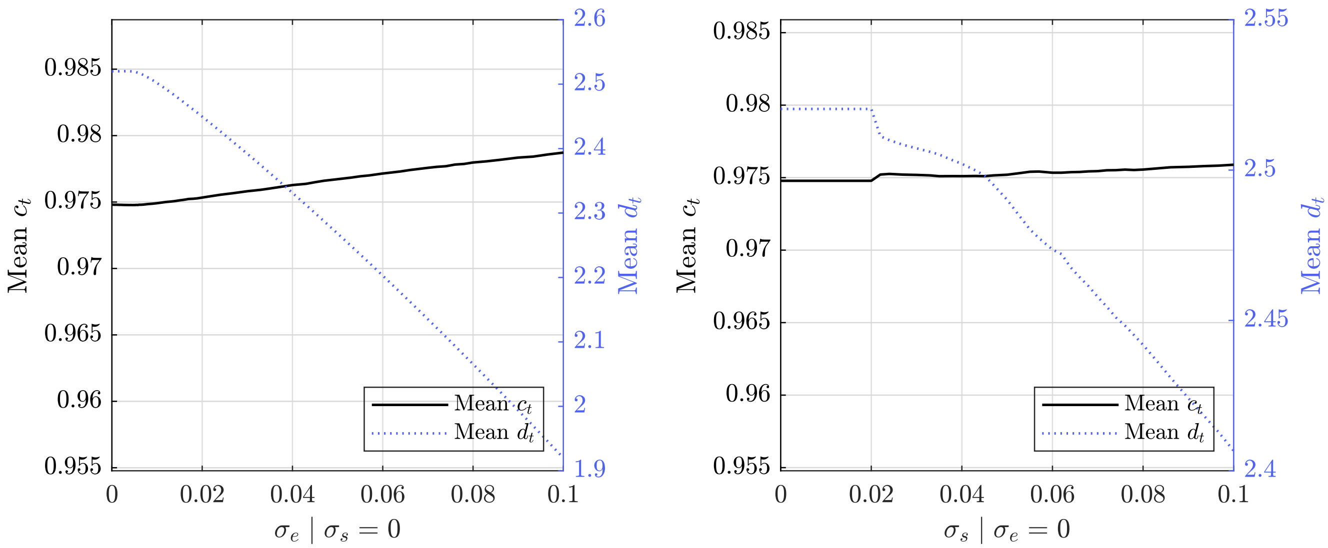

Figure I.3: Varying uncertainty in the economy with a nonbinding steady state. In each panel, the continuous (black) line reports the pattern of average consumption, and the dotted (blue) line that of average debt for different standard deviations of a given shock, conditional on the other shock being switched off. All the other parameters are at their baseline values.