Additional Appendix to

"Deepening Contractions and Collateral Constraints"

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This appendix contains further details about the model and its solution.

1 The steady state

The deterministic steady state of our model is described in the following, where variables without time subscripts are the steady-state values. We first consider the implications of the patient households' optimality conditions, from which we get

$$\frac{1-\beta^P\rho^P}{(1-\rho^P)\,C^P}=\lambda^P, \tag{1}$$

and

$$\nu^P\left(1-N^P\right)^{-\varphi^P}=\lambda^P W^P, \tag{2}$$

respectively. The steady-state gross interest rate on loans is obtained from the Euler equation:

$$\begin{aligned} \beta^P R\lambda^P &= \lambda^P, \\ R &= \frac{1}{\beta^P}, \end{aligned} \tag{3}$$

emphasizing that it is the time preferences of the most patient individual that determine the steady-state rate of interest. Further, we get

$$\begin{aligned} \frac{\varepsilon}{H^P}+\beta^P\lambda^P Q &= \lambda^P Q, \\ H^P &= \frac{\varepsilon}{Q\lambda^P\left(1-\beta^P\right)}. \end{aligned} \tag{4}$$

Turning to the impatient households, we obtain

$$\frac{1-\beta^I\rho^I}{(1-\rho^I)\,C^I}=\lambda^I, \tag{5}$$

and

$$\nu^I \left(1 - N^I\right)^{-\varphi^I} = \lambda^I W^I, \tag{6}$$

respectively. From the Euler equation, we obtain the steady-state value of the multiplier on the credit constraint:

$$\mu^I = \lambda^I \left(1 - \beta^I R\right),$$

which by use of (3) yields

$$\mu^I = \lambda^I \left(1 - \frac{\beta^I}{\beta^P}\right). \tag{7}$$

From (7), we see that in steady state $\mu^I > 0$ since $\beta^P > \beta^I$, which proves that the credit constraint is binding in steady state. In a similar fashion, we get from the entrepreneur's Euler equation:

$$\mu^E = \lambda^E \left(1 - \frac{\beta^E}{\beta^P}\right). \tag{8}$$

Hence, $\mu^E > 0$ implying that the entrepreneurs' credit constraints are also binding in steady state. For the impatient household's optimality conditions, we further get:

$$\begin{aligned}
\frac{\varepsilon}{H^I} + \beta^I \lambda^I Q + \mu^I s \frac{Q}{R} &= \lambda^I Q, \\
H^I &= \frac{\varepsilon}{Q\lambda^I \left[1 - \beta^I - \frac{\mu^I}{\lambda^I} s \frac{1}{R}\right]}, \\
H^I &= \frac{\varepsilon}{Q\lambda^I \left[1 - \beta^I - \left(1 - \frac{\beta^I}{\beta^P}\right) s\beta^P\right]}, \\
H^I &= \frac{\varepsilon}{Q\lambda^I \left[1 - \beta^I - \left(\beta^P - \beta^I\right) s\right]},
\end{aligned} \tag{9}$$

where we have made use of (3) and (7).

Turning to the remaining optimality conditions of the entrepreneurs, we get

$$\frac{1 - \beta^E \rho^E}{\left(1 - \rho^E\right) C^E} = \lambda^E, \tag{10}$$

and

$$\psi^E \left[1 - \frac{\Omega}{2}\left(\frac{I}{I} - 1\right)^2\right] - \psi^E \Omega \frac{I}{I}\left(\frac{I}{I} - 1\right) + \beta^E \psi^E \Omega \left(\frac{I}{I}\right)^2 \left(\frac{I}{I} - 1\right) = \lambda^E,$$

leading to

$$\psi^E = \lambda^E. \tag{11}$$

This reflects that there are no investment adjustment costs in steady state in our no-growth model. Therefore, the shadow value of a unit of capital equals the shadow value of wealth. Combining this with $Q_t^K = \psi_t^E / \lambda_t^E$, we readily obtain

$$Q^K = 1. \tag{12}$$

From the optimality condition for capital, we obtain

$$\begin{aligned} \beta^E r^K \lambda^E + \beta^E (1-\delta)\psi^E + \mu^E s \frac{Q^K}{R} &= \psi^E \\ \beta^E \lambda^E r^K + \beta^E (1-\delta)\psi^E + \lambda^E \left(1 - \frac{\beta^E}{\beta^P}\right) s \frac{Q^K}{R} &= \psi^E \\ 1 + r^K - \delta &= \frac{1 - \left(\beta^P - \beta^E\right) s Q^K}{\beta^E}, \end{aligned} \tag{13}$$

where the second line uses (8), and the last uses (3) and (11), respectively. Likewise, from the housing optimality condition we find:

$$r^H = \frac{\left(1-\beta^E\right) Q}{\beta^E} - \frac{\mu^E s}{\lambda^E \beta^E} \frac{Q}{R}. \tag{14}$$

We then turn to the remaining equilibrium conditions in steady state. As we saw above, the two credit constraints are binding in steady state. Hence,

$$B^I = \frac{sQH^I}{R}, \tag{15}$$

$$B^E = s \frac{Q^K K + Q H^E}{R}. \tag{16}$$

The production function is

$$Y = \left[\left(N^P\right)^\alpha \left(N^I\right)^{1-\alpha}\right]^\gamma \left[\left(H^E\right)^\phi K^{1-\phi}\right]^{1-\gamma}. \tag{17}$$

The steady-state versions of the firms' first-order conditions taking market clearing conditions into account are

$$\begin{aligned}
\alpha\gamma\frac{Y}{N^P} &= W^P, && (18)\\
(1-\alpha)\,\gamma\frac{Y}{N^I} &= W^I, && (19)\\
(1-\gamma)\,(1-\phi)\,\frac{Y}{K} &= r^K, && (20)\\
(1-\gamma)\,\phi\frac{Y}{H^E} &= r^H. && (21)
\end{aligned}$$

In steady state the law of motion for capital implies

$$I = \delta K. \tag{22}$$

We have the following steady-state resource constraints:

$$Y = C^P + C^I + C^E + I, \tag{23}$$

$$H = H^P + H^I + H^E, \tag{24}$$

$$B^P + B^I + B^E = 0. \tag{25}$$

Also, we have the steady-state versions of the agents' budget constraints:

$$\begin{aligned}
C^P &= W^P N^P - (R-1)\,B^P, && (26)\\
C^I &= W^I N^I - (R-1)\,B^I, && (27)\\
C^E + I &= r^K K + r^H H^E - (R-1)\,B^E && (28)
\end{aligned}$$

(One of these is redundant by Walras' law.)

We therefore have that the steady state is characterized by the vector

$$\left[\begin{array}{c} Y, C^P, C^I, C^E, I, H^P, H^I, H^E, K, N^P, N^I, B^P, B^I, B^E, \\ Q, Q^K, R, r^K, r^H, W^P, W^I, \lambda^P, \lambda^I, \lambda^E, \mu^I, \mu^E, \psi^E \end{array}\right].$$

These 27 variables are determined by the 27 equations (1), (2), (3), (4), (5), (6), (7), (8), (9), (10), (11), (13), (14), (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (26), (27), and (12).

We now briefly proceed with a characterization of the steady state, wherein we compute some variables in ratios to output in closed form. Then we reduce the system to one of seven equations in central quantities, which is solved numerically, conditional on these ratios. The remaining 19 variables then follow explicitly from the characterizations given above. First, combine (20) and (13) to get an expression for capital-output ratio:

$$\frac{n_E K}{nY} = \frac{\beta^E (1-\gamma)(1-\phi)}{1-\left(\beta^P - \beta^E\right) s - \beta^E (1-\delta)}, \tag{29}$$

where we have used that $Q^K = 1$ by (12), Then we combine (14) and (21) to get an expression for entrepreneurs' housing-output ratio:

$$\begin{aligned}
(1-\gamma)\phi \frac{Y}{H^E} &= \frac{\left(1-\beta^E\right) Q}{\beta^E} - \frac{\mu^E s}{\lambda^E \beta^E}\frac{Q}{R}, \\
\frac{Y}{H^E} &= \frac{\left(1-\beta^E\right) Q \lambda^E R - \mu^E s Q}{(1-\gamma)\phi\beta^E\lambda^E R}, \\
\frac{QH^E}{Y} &= \frac{(1-\gamma)\phi\beta^E}{\left(1-\beta^E\right) - \left(\beta^P-\beta^E\right) s},
\end{aligned} \tag{30}$$

where we have used (8). Again using that $Q^K = 1$, the borrowing constraint for entrepreneurs (16) can be rewritten in terms of ratios to output as

$$\frac{B^E}{Y} = \frac{s}{R}\left(\frac{K}{Y} + \frac{QH^E}{Y}\right),$$

which by use of (29), (30) and (3) implies

$$\frac{B^E}{Y} = \beta^P s \left(\frac{\beta^E (1-\gamma)(1-\phi)}{1-\left(\beta^P-\beta^E\right) s - \beta^E (1-\delta)} + \frac{(1-\gamma)\phi\beta^E}{\left(1-\beta^E\right) - \left(\beta^P-\beta^E\right) s}\right) \tag{31}$$

This closed-form solution of the entrepreneurs' steady-state loan-to-output ratio is central in setting up a subsystem of seven central variables. First, it can be used with the entrepreneur's budget constraint, (28), in ratio to output:

$$\frac{C^E}{Y} + \frac{I}{Y} = r^K \frac{K}{Y} + r^H \frac{H^E}{Y} - (R-1)\frac{B^E}{Y},$$

which by use of (22) becomes

$$\frac{C^E}{Y} = \left(r^K - \delta\right)\frac{K}{Y} + r^H \frac{H^E}{Y} - (R-1)\frac{B^E}{Y}.$$

Using (13) and (21) we get

$$\frac{C^E}{Y} = \left(\frac{1-\beta^E - \left(\beta^P - \beta^E\right)s}{\beta^E}\right)\frac{K}{Y} + (1-\gamma)\,\phi - (R-1)\frac{B^E}{Y},$$

which by use of (29) provides the entrepreneurs' consumption-to-output ratio:

$$\frac{C^E}{Y} = \frac{(1-\gamma)(1-\phi)\left[1-\beta^E - \left(\beta^P - \beta^E\right)s\right]}{1-\left(\beta^P - \beta^E\right)s - \beta^E(1-\delta)} + (1-\gamma)\,\phi - \frac{1-\beta^P}{\beta^P}\frac{B^E}{Y}, \tag{32}$$

Then turn to the impatient households. In ratio to output, their budget constraints are, cf. (27),

$$\frac{C^I}{Y} = \frac{W^I N^I}{Y} - (R-1)\frac{B^I}{Y},$$

which by use of (19) and (3) becomes

$$\frac{C^I}{Y} = (1-\alpha)\,\gamma - \frac{1-\beta^P}{\beta^P}\frac{B^I}{Y}.$$

Likewise, the patient households' budget constraints are written as, cf. (26),

$$\frac{C^P}{Y} = \frac{W^P N^P}{Y} - (R-1)\frac{B^P}{Y},$$

which by use of (18) and (3) becomes

$$\frac{C^P}{Y} = \alpha\gamma - \frac{1-\beta^P}{\beta^P}\frac{B^P}{Y}.$$

Adding these constraints gives

$$\frac{C^I + C^P}{Y} = \gamma + \frac{1-\beta^P}{\beta^P}\frac{B^E}{Y}, \tag{33}$$

where (25) has been invoked. Note that the right-hand-side of (33) is known by (31).

Combining (1), (2) and (18) gives the steady-state equilibrium condition for the labor market for impatient households:

$$\nu^P \left(1 - N^P\right)^{-\varphi^P} C^P \frac{1-\rho^P}{1-\beta^P \rho^P} = \alpha\gamma \frac{Y}{N^P}. \tag{34}$$

Similarly, (5), (6) and (19) characterize the labor-market equilibrium for impatient households:

$$\nu^I \left(1 - N^I\right)^{-\varphi^I} C^I \frac{1-\rho^I}{1-\beta^I \rho^I} = (1-\alpha)\,\gamma \frac{Y}{N^I}. \tag{35}$$

Combining the two households' land demand expressions, (4) and (9), gives

$$\frac{H^I}{H^P} = \frac{\lambda^P \left(1-\beta^P\right)}{\lambda^I \left[1 - \beta^I - \left(\beta^P - \beta^I\right) s\right]}.$$

Eliminating the multipliers by (1) and (5), and eliminating H^P by (24), we obtain the following land-market equilibrium characterization:

$$\begin{aligned}
\frac{H^I}{H - H^I - H^E} &= \frac{\frac{1-\beta^P\rho^P}{(1-\rho^P)C^P}\left(1-\beta^P\right)}{\frac{1-\beta^I\rho^I}{(1-\rho^I)C^I}\left[1 - \beta^I - \left(\beta^P - \beta^I\right) s\right]}, \\
\frac{H^I}{H - H^I - H^E}\frac{C^P}{C^I} &= \frac{\left(1-\beta^P\rho^P\right)\left(1-\rho^I\right)}{\left(1-\rho^P\right)\left(1-\beta^I\rho^I\right)} \frac{\left(1-\beta^P\right)}{\left[1 - \beta^I - \left(\beta^P - \beta^I\right) s\right]}.
\end{aligned} \tag{36}$$

We also take the impatient households' borrowing constraint into consideration. Using (15) to eliminate B^I in the budget constraint, it becomes

$$\begin{aligned}
\frac{C^I}{Y} &= (1-\alpha)\,\gamma - \frac{1-\beta^P}{\beta^P}\frac{sQH^I}{YR}, \\
&= (1-\alpha)\,\gamma - \left(1-\beta^P\right)\frac{sQH^I}{Y}.
\end{aligned} \tag{37}$$

We can use that (9) implies

$$QH^I = \frac{\varepsilon}{\lambda^I \left[1 - \beta^I - \left(\beta^P - \beta^I\right) s\right]}$$

and thus, again using (5),

$$QH^I = \frac{\frac{\varepsilon\left(1-\rho^I\right)}{1-\beta^I\rho^I}C^I}{1-\beta^I-\left(\beta^P-\beta^I\right)s}, \tag{38}$$

$$Q = \frac{\frac{\varepsilon\left(1-\rho^I\right)}{1-\beta^I\rho^I}C^I}{H^I\left[1-\beta^I-\left(\beta^P-\beta^I\right)s\right]}. \tag{39}$$

We then use (38) to rewrite the consumption-output ratio for impatient households (37) as:

$$\begin{aligned} \frac{C^I}{Y} &= (1-\alpha)\gamma - \left(1-\beta^P\right)\frac{sQH^I}{Y} \\ &= (1-\alpha)\gamma - \left(1-\beta^P\right)\frac{s}{Y}\frac{\frac{\varepsilon\left(1-\rho^I\right)}{1-\beta^I\rho^I}C^I}{1-\beta^I-\left(\beta^P-\beta^I\right)s}. \end{aligned} \tag{40}$$

Likewise, we rewrite the entrepreneurs' land to output ratio by using (39) to eliminate Q from (30):

$$\frac{H^E}{Y} = \frac{(1-\gamma)\phi\beta^E}{1-\beta^E-\left(\beta^P-\beta^E\right)s}\frac{H^I\left[1-\beta^I-\left(\beta^P-\beta^I\right)s\right]}{\frac{\varepsilon(1-\rho^I)}{1-\beta^I\rho^I}C^I}. \tag{41}$$

Finally, the production function (17) is rewritten as a function of the derived ratios:

$$Y^\gamma = A\left[\left(N^P\right)^\alpha\left(N^I\right)^{1-\alpha}\right]^\gamma\left[\left(\frac{H^E}{Y}\right)^\phi\left(\frac{K}{Y}\right)^{1-\phi}\right]^{1-\gamma},$$

Using (29), we finally obtain

$$Y = A^{\frac{1}{\gamma}}\left(N^P\right)^\alpha\left(N^I\right)^{1-\alpha}\left[\left(\frac{H^E}{Y}\right)^\phi\left(\frac{\beta^E(1-\gamma)(1-\phi)}{1-\left(\beta^P-\beta^E\right)s-\beta^E(1-\delta)}\right)^{1-\phi}\right]^{\frac{1-\gamma}{\gamma}}. \tag{42}$$

We have now reduced the steady-state to a matter of finding the vector

$$\left[Y, C^P, C^I, H^I, H^E, N^P, N^I\right],$$

which satisfies the equations (33), (34), (35), (36), (40), (41) and (42), *given* the solution for $n_E B^E/(nY)$, (31), and given all parameters and exogenous variables of the model. We compute the vector numerically using `fsolve` in Matlab. The remaining 19 variables then follow analytically from the steady-state equations presented above.

2 The log-linearized model

We log-linearize the model around the steady state found in the previous section. In the following, we let $\widehat{X_t}$ denote the log-deviation of a generic variable X_t from its steady state value X, except for the following variables. For the interest rates, $\widehat{R}_t \equiv R_t - R$, $\widehat{r}_t^H \equiv r_t^H - r^H$ and $\widehat{r}_t^K \equiv r_t^K - r^K$, and for debt, $\widehat{B}_t^i \equiv n_i \left(B_t^i - B^i\right) / \left(nY\right)$, $i = P, I, E$. We first derive the log-linear versions of the agents' optimality conditions and conclude with the expressions for market clearing.

2.1 Optimality conditions of the patient households

We obtain the following:

$$\beta^P \rho^P \mathrm{E}_t \left\{ \widehat{C}_{t+1}^P \right\} - \left(1 + \beta^P \left(\rho^P\right)^2\right) \widehat{C}_t^P + \rho^P \widehat{C}_{t-1}^P = \left(1 - \rho^P\right)\left(1 - \beta^P \rho^P\right) \widehat{\lambda}_t^P, \qquad (43)$$

$$\varphi^P \frac{N^P}{1 - N^P} \widehat{N}_t^P = \widehat{\lambda}_t^P + \widehat{W}_t^P \qquad (44)$$

$$\beta^P \widehat{R}_t + \mathrm{E}_t \left\{ \widehat{\lambda}_{t+1}^P \right\} = \widehat{\lambda}_t^P, \qquad (45)$$

$$\frac{\varepsilon}{H^P}\left(\widehat{\varepsilon}_t - \widehat{H}_t^P\right) + \beta^P \lambda^P Q \mathrm{E}_t \left\{ \widehat{\lambda}_{t+1}^P + \widehat{Q}_{t+1} \right\} = \lambda^P Q \left(\widehat{\lambda}_t^P + \widehat{Q}_t\right).$$

which, by use of steady-state equation (4) becomes

$$-Q\lambda^P \left(1 - \beta^P\right) \widehat{H}_t^P + Q\lambda^P \left(1 - \beta^P\right) \widehat{\varepsilon}_t + \beta^P \lambda^P Q \mathrm{E}_t \left\{ \widehat{\lambda}_{t+1}^P + \widehat{Q}_{t+1} \right\} = \lambda^P Q \left(\widehat{\lambda}_t^P + \widehat{Q}_t\right),$$

and thereby

$$\beta^P \mathrm{E}_t \left\{ \widehat{\lambda}_{t+1}^P + \widehat{Q}_{t+1} \right\} - \left(1 - \beta^P\right) \widehat{H}_t^P + \left(1 - \beta^P\right) \widehat{\varepsilon}_t = \widehat{\lambda}_t^P + \widehat{Q}_t. \qquad (46)$$

Moreover, the log-linearized budget constraint holds:

$$\begin{aligned} & \frac{C^P}{Y} \widehat{C}_t^P + \frac{QH^P}{Y}\left(\widehat{H}_t^P - \widehat{H}_{t-1}^P\right) + \frac{B^P}{Y} \widehat{R}_{t-1} + \frac{1}{\beta^P} \widehat{B}_{t-1}^P \\ = \; & \widehat{B}_t^P + \alpha\gamma \left(\widehat{W}_t^P + \widehat{N}_t^P\right). \end{aligned}$$

where we have used (18). This constraint, however, does not feature in our MATLAB codes (we use the impatient households' and entrepreneurs' budget constraint and the economy-wide resource constraint).

2.2 Optimality conditions of the impatient households

For the impatient households, we obtain

$$\beta^I \rho^I \mathrm{E}_t \left\{ \widehat{C}^I_{t+1} \right\} - \left(1 + \beta^I \left(\rho^I\right)^2\right) \widehat{C}^I_t + \rho^I \widehat{C}^I_{t-1} = \left(1 - \rho^I\right)\left(1 - \beta^I \rho^I\right) \widehat{\lambda}^I_t, \tag{47}$$

$$\varphi^I \frac{N^I}{1 - N^I} \widehat{N}^I_t = \widehat{\lambda}^I_t + \widehat{W}^I_t, \tag{48}$$

and

$$\beta^I \lambda^I \widehat{R}_t + \beta^I R \lambda^I \mathrm{E}_t \left\{ \widehat{\lambda}^I_{t+1} \right\} + \mu^I \widehat{\mu}^I_t = \lambda^I \widehat{\lambda}^I_t,$$

respectively. The last expression is rewritten by use of (7):

$$\beta^I \widehat{R}_t + \beta^I R \mathrm{E}_t \left\{ \widehat{\lambda}^I_{t+1} \right\} + \left(1 - \frac{\beta^I}{\beta^P}\right) \widehat{\mu}^I_t = \widehat{\lambda}^I_t. \tag{49}$$

Furthermore, we get:

$$\begin{aligned} & \frac{\varepsilon}{H^I}\left(\widehat{\varepsilon}_t - \widehat{H}^I_t\right) + \beta^I \lambda^I Q \mathrm{E}_t \left\{ \widehat{\lambda}^I_{t+1} + \widehat{Q}_{t+1} \right\} \\ & + \mu^I \frac{sQ}{R}\left[\widehat{\mu}^I_t + \widehat{s}_t + \mathrm{E}_t\left\{\widehat{Q}_{t+1}\right\} - \beta^P \widehat{R}_t\right] \\ = \; & \lambda^I Q \left(\widehat{\lambda}^I_t + \widehat{Q}_t\right), \end{aligned}$$

which by use of (7) and (9) becomes

$$\begin{aligned} & Q\lambda^I \left[1 - \beta^I - \left(\beta^P - \beta^I\right) s\right]\left(\widehat{\varepsilon}_t - \widehat{H}^I_t\right) + \beta^I \lambda^I Q \mathrm{E}_t \left\{ \widehat{\lambda}^I_{t+1} + \widehat{Q}_{t+1} \right\} \\ & + \lambda^I \left(1 - \frac{\beta^I}{\beta^P}\right) \frac{sQ}{R}\left[\widehat{\mu}^I_t + \widehat{s}_t + \mathrm{E}_t\left\{\widehat{Q}_{t+1}\right\} - \beta^P \widehat{R}_t\right] \\ = \; & \lambda^I Q \left(\widehat{\lambda}^I_t + \widehat{Q}_t\right), \end{aligned}$$

$$
\begin{aligned}
& \beta^I \mathrm{E}_t \left\{ \widehat{\lambda}^I_{t+1} + \widehat{Q}_{t+1} \right\} - \left[1 - \beta^I - s \left(\beta^P - \beta^I \right) \right] \widehat{H}^I_t \\
& + \left[1 - \beta^I - \left(\beta^P - \beta^I \right) s \right] \widehat{\varepsilon}_t \\
& + s \left(\beta^P - \beta^I \right) \left[\widehat{\mu}^I_t + \widehat{s}_t + \mathrm{E}_t \left\{ \widehat{Q}_{t+1} \right\} - \beta^P \widehat{R}_t \right] \\
= \; & \widehat{\lambda}^I_t + \widehat{Q}_t. \qquad (50)
\end{aligned}
$$

where we have again used (3). The budget constraint becomes

$$
\frac{C^I}{Y} \widehat{C}^I_t + \frac{QH^I}{Y} \left(\widehat{H}^I_t - \widehat{H}^I_{t-1} \right) + \frac{B^I}{Y} \widehat{R}_{t-1} + \frac{1}{\beta^P} \widehat{B}^I_{t-1} = \widehat{B}^I_t + (1 - \alpha) \gamma \left(\widehat{W}^I_t + \widehat{N}^I_t \right), \quad (51)
$$

where we have used (19). Finally, the log-linearized version of the collateral constraint is:

$$
\frac{Y}{B^I} \widehat{B}^I_t \leq \hat{s}_t + \mathrm{E}_t \left\{ \widehat{Q}_{t+1} \right\} + \widehat{H}^I_t - \beta^P \widehat{R}_t. \quad (52)
$$

Note that while the credit constraint binds in steady state, cf. (15), we allow it to be non-binding outside steady state.

2.3 Optimality conditions of the entrepreneurs

We have the following conditions:

$$
\beta^E \rho^E \mathrm{E}_t \left\{ \widehat{C}^E_{t+1} \right\} - \left(1 + \beta^E \left(\rho^E \right)^2 \right) \widehat{C}^E_t + \rho^E \widehat{C}^E_{t-1} = \left(1 - \rho^E \right) \left(1 - \beta^E \rho^E \right) \widehat{\lambda}^E_t, \quad (53)
$$

$$
\beta^E \widehat{R}_t + \beta^E R \mathrm{E}_t \left\{ \widehat{\lambda}^E_{t+1} \right\} + \left(1 - \frac{\beta^E}{\beta^P} \right) \widehat{\mu}^E_t = \widehat{\lambda}^E_t. \quad (54)
$$

$$
\widehat{\psi}^E_t - \Omega \left(1 + \beta^E \right) \widehat{I}_t + \Omega \widehat{I}_{t-1} + \beta^E \Omega \mathrm{E}_t \left\{ \widehat{I}_{t+1} \right\} = \widehat{\lambda}^E_t, \quad (55)
$$

where we have made use of (11). Furthermore:

$$
\begin{aligned}
& \beta^E \widehat{r}^K_t + \beta^E r^K \mathrm{E}_t \left\{ \widehat{\lambda}^E_{t+1} \right\} + (1 - \delta) \beta^E \mathrm{E}_t \left\{ \widehat{\psi}^E_{t+1} \right\} \\
& + \left(\beta^P - \beta^E \right) s Q^K \left(\widehat{\mu}^E_t + \widehat{s}_t + \mathrm{E}_t \left\{ \widehat{Q}^K_{t+1} \right\} - \beta^P \widehat{R}_t \right) \\
= \; & \widehat{\psi}^E_t \qquad (56)
\end{aligned}
$$

where we have used (11) and (3). Moreover, we have

$$\widehat{\psi}_t^E = \widehat{\lambda}_t^E + \widehat{Q}_t^K, \tag{57}$$

and

$$\begin{aligned} & \beta^E Q \left(\mathrm{E}_t \left\{ \widehat{\lambda}_{t+1}^E \right\} + \mathrm{E}_t \left\{ \widehat{Q}_{t+1} \right\} \right) + \beta^E r^H \left(\mathrm{E}_t \left\{ \widehat{\lambda}_{t+1}^E \right\} + \frac{1}{r^H} \widehat{r}_t^H \right) \\ & + \left(\beta^P - \beta^E \right) sQ \left(\widehat{\mu}_t^E + \widehat{s}_t + \mathrm{E}_t \left\{ \widehat{Q}_{t+1} \right\} - \beta^P \widehat{R}_t \right) \\ = \; & Q \left(\widehat{\lambda}_t^E + \widehat{Q}_t \right). \end{aligned} \tag{58}$$

The budget constraint for entrepreneurs becomes

$$\begin{aligned} & \frac{C^E}{Y} \widehat{C}_t^E + \frac{I}{Y} \widehat{I}_t + \frac{QH^E}{Y} \left(\widehat{H}_t^E - \widehat{H}_{t-1}^E \right) + \frac{B^E}{Y} \widehat{R}_{t-1} + \frac{1}{\beta^P} \widehat{B}_{t-1}^E \\ = \; & \widehat{B}_t^E + \frac{K}{Y} \widehat{r}_{t-1}^K + \frac{H^E}{Y} \widehat{r}_{t-1}^H + (1-\gamma) \phi \widehat{H}_{t-1}^E + (1-\gamma)(1-\phi) \widehat{K}_{t-1}. \end{aligned} \tag{59}$$

where we have used (20) and (21). The borrowing constraint must be satisfied:

$$\begin{aligned} Y \widehat{B}_t^E \; \leq \; & s \frac{\left(K + QH^E \right)}{R} \widehat{s}_t - \frac{s}{R^2} \left(K + QH^E \right) \widehat{R}_t + \frac{sK}{R} \mathrm{E}_t \left\{ \widehat{Q}_{t+1}^K \right\} \\ & + \frac{sK}{R} \widehat{K}_t + \frac{sQH^E}{R} \mathrm{E}_t \left\{ \widehat{Q}_{t+1} \right\} + \frac{sQH^E}{R} \widehat{H}_t^E. \end{aligned}$$

Dividing by the steady-state values on both sides:

$$\begin{aligned} \frac{Y}{B^E} \widehat{B}_t^E \; \leq \; & \widehat{s}_t - \beta^P \widehat{R}_t \\ & + \frac{K}{K + QH^E} \mathrm{E}_t \left\{ \widehat{Q}_{t+1}^K \right\} + \frac{K}{K + QH^E} \widehat{K}_t + \frac{QH^E}{K + QH^E} \mathrm{E}_t \left\{ \widehat{Q}_{t+1} \right\} + \frac{QH^E}{K + QH^E} \widehat{H}_t^E, \end{aligned}$$

yielding

$$\frac{Y}{B^E} \widehat{B}_t^E \leq \widehat{s}_t - \beta^P \widehat{R}_t + \frac{K}{K + QH^E} \left(\mathrm{E}_t \left\{ \widehat{Q}_{t+1}^K \right\} + \widehat{K}_t \right) + \frac{QH^E}{K + QH^E} \left(\mathrm{E}_t \left\{ \widehat{Q}_{t+1} \right\} + \widehat{H}_t^E \right). \tag{60}$$

2.4 Optimality conditions of the firms

The first-order conditions of firms are readily rewritten as

$$\widehat{Y}_t - \widehat{N}_t^P = \widehat{W}_t^P, \tag{61}$$

$$\widehat{Y}_t - \widehat{N}_t^I = \widehat{W}_t^I, \tag{62}$$

$$\mathrm{E}_t \left\{ \widehat{Y}_{t+1} \right\} - \widehat{K}_t = \left(r^K \right)^{-1} \widehat{r}_t^K, \tag{63}$$

$$\mathrm{E}_t \left\{ \widehat{Y}_{t+1} \right\} - \widehat{H}_t^E = \left(r^H \right)^{-1} \widehat{r}_t^H, \tag{64}$$

respectively.

2.5 Market clearing and resource constraints

From the law of motion for capital, we get:

$$\widehat{K}_t = (1 - \delta) \widehat{K}_{t-1} + \delta \widehat{I}_t. \tag{65}$$

where we have used (22). Moreover, the aggregate resource constraint gives:

$$\widehat{Y}_t = \frac{C^P}{Y} \widehat{C}_t^P + \frac{C^I}{Y} \widehat{C}_t^I + \frac{C^E}{Y} \widehat{C}_t^E + \delta \frac{K}{Y} \widehat{I}_t \tag{66}$$

We also have the linearized versions of the production function and the housing and bond market clearing conditions:

$$\widehat{Y}_t = \widehat{A}_t + \alpha \gamma \widehat{N}_t^P + (1 - \alpha) \gamma \widehat{N}_t^I + (1 - \gamma)(1 - \phi) \widehat{K}_{t-1} + (1 - \gamma) \phi \widehat{H}_{t-1}^E \tag{67}$$

$$0 = H^P \widehat{H}_t^P + H^I \widehat{H}_t^I + H^E \widehat{H}_t^E \tag{68}$$

$$0 = \widehat{B}_t^P + \widehat{B}_t^I + \widehat{B}_t^E \tag{69}$$

Finally, we have the shock processes. For the technology shock, we get:

$$\widehat{A}_t = \rho_A \widehat{A}_{t-1} + z_t. \tag{70}$$

Likewise, the housing preference shock becomes

$$\widehat{\varepsilon}_t = \rho_\varepsilon \widehat{\varepsilon}_{t-1} + u_t, \tag{71}$$

while the credit limit shock gives

$$\widehat{s}_t = \rho_s \widehat{s}_{t-1} + v_t, \tag{72}$$

which completes our list of log-linearized equations.

The log-linearized system consists of 30 equations: 18 first-order conditions, 2 budget constraints, 2 credit constraints, 1 production function, 3 market clearing conditions, 1 capital accumulation equation, and 3 shock processes. The 30 variables of the system are given by the vector

$$\left[\begin{array}{c} \widehat{C}_t^P, \widehat{C}_t^I, \widehat{C}_t^E, \widehat{\lambda}_t^P, \widehat{\lambda}_t^I, \widehat{\lambda}_t^E, \widehat{\psi}_t^E, \widehat{\mu}_t^I, \widehat{\mu}_t^E, \widehat{R}_t, \widehat{N}_t^P, \widehat{N}_t^I, \widehat{W}_t^P, \widehat{W}_t^I, \\ \widehat{H}_t^P, \widehat{H}_t^I, \widehat{H}_t^E, \widehat{Q}_t, \widehat{Q}_t^K, \widehat{r}_t^H, \widehat{r}_t^K, \widehat{K}_t, \widehat{I}_t, \widehat{Y}_t, \widehat{B}_t^P, \widehat{B}_t^I, \widehat{B}_t^E, \widehat{A}_t, \widehat{\varepsilon}_t, \widehat{s}_t \end{array}\right],$$

and is characterized by equations (43)-(72).